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A STUDY ON LEBESGUE DECOMPOSITION

OF MEASURES INDUCED BY

STABLE PROCESSES

by

Mauro Sergio de Freitas Marques

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MAURO SERGIO DE FREITAS MARQUES. A Study on Lebesgue Decomposition of Measures Induced by Stable Processes. (Under the direction of STAMATIS CAMBANIS)

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CHAPTER I

INTRODUCTION

1.1 Introduction

The Lebesgue decomposition of measures induced by stochastic processes is important in areas such as statistical inference and information theory.

For Gaussian processes the Lebesgue decomposition has been fully described and the following dichotomy prevails: two Gaussian processes are either mutually absolutely continuous, or else they are singular (see, e.g. Chatterji and Mandrekar (1978)). In the former case expressions for the Radon-Nikodym derivative are known and the discrimination of the two Gaussian processes is based on a threshold test on the log of their likelihood ratio. In the latter case they can in principle be discriminated with probability one. Some partial results are also available for other processes having finite second moment (Fortet (1973)).

The Central Limit Theorem and the stability property provide the basic reasons for regarding stable processes as a natural generalization of Gaussian processes. Most of the work on stable processes focuses on contrasts and similarities between Gaussian and non-Gaussian stable processes. While the problem of Lebesgue decomposition of measures induced by Gaussian processes is the simplest and most thoroughly studied for non-Gaussian stable processes the problem has remained largely open.

This work investigates mainly the Lebesgue decomposition of measures induced by non-Gaussian stable processes. For non-Gaussian measures, this question seems to have been first studied by Gihman and Skorohod (1966) and Skorohod (1965) for infinitely divisible measures in Hilbert space, and subsequently by Briggs (1975), Veeh (1981), and

Brockett (1984) for measures induced by infinitely divisible processes. Except for the first work no application of the results to non-Gaussian stable measures has been made. The only works dealing specifically with stable measures are Zinn (1975) and Thang and Tien (1980).

Sufficient conditions for an element to be an admissible translate of an infinitely divisible measure in a Hilbert space were obtained in Gihman and Skorohod (1966). However, as observed by Zinn (1975), these conditions are difficult to verify and, as simplified for stable measures, they were found to be false.

Zinn (1976), investigated the structure of the set of admissible translates of stable measures in a Hilbert space. As an application he showed that certain stable processes have no nontrivial admissible translates. The admissible translates of symmetric stable measures with discrete spectral measures in a Banach space were characterized by Thang and Tien (1980).

All these works use primarily the representation of the characteristic functional of a stable measure in Hilbert or Banach space. Here we work with stable processes and exploit their spectral representation, which in some cases allows the formulation of the problem in terms of processes with independent increments and/or sequences of independent random variables.

1.2 Summary

The next section of this chapter (1.3) introduces the setting and notation, and presents the basic definitions and results on stable processes.

Chapter II considers the Lebesgue decomposition between the measure induced by a stochastic process and its translates by a nonrandom function, i.e. the problem of detecting a nonrandom signal in additive random noise. In Section 2.1, for p^{th} order and symmetric stable processes a function space is introduced which plays a role partly analogous to the reproducing kernel Hilbert space of a Gaussian or second order process.

In particular this space provides an upper bound for the set of admissible translates, is a stochastic processes version of a space introduced by Zinn (1975), page 249, and extends the results of Zinn (1975), Proposition 10, to general symmetric stable processes and the results of Fortet (1973), Théorème 4.1, to general p^{th} order processes. A lower bound for the set of admissible translates of a stable process is also provided by exploiting their structure as mixtures of Gaussian processes, and a dichotomy is shown for a class of stable processes which includes all sub-Gaussian and sub-Gaussian-like processes.

In Section 2 of Chapter II, stable processes with an invertible spectral representation are considered. Their admissible translates are characterized, and a dichotomy is established: each translate is either admissible or singular. The result is applied to show that most continuous time moving averages, and all harmonizable processes with nonatomic spectral measure have no admissible translate. Thus these processes do not provide realistic models for additive noise, as every nonrandom signal can be perfectly detected in their presence. General harmonizable processes and discrete time mixed autoregressive moving averages processes are also considered.

Section 3, Chapter II, comments on the Radon-Nikodym derivatives in the case of an admissible translate and provides an expression for the likelihood ratio in terms of the one dimensional stable density in the case of purely atomic control measures.

Chapter III considers the Lebesgue decomposition between two measures induced by certain non-Gaussian processes. In Section 3.1 equivalence and singularity of product measures are studied. An idea of LeCam (1970) is developed further and provides a necessary and sufficient condition for equivalence and for singularity of certain product measures. As an application, the results of Steele (1986) on the discrimination between a sequence of random vectors in $\mathbf{R}^k$ and its perturbation by rigid motions, are extended to more general classes of perturbations; and for certain non-symmetric (skewed) stable sequences of independent random variables, necessary and sufficient conditions are given for equivalence and for singularity. The singularity between sequences of independent

symmetric stable random variables with different indexes of stability is also proved.

Section 2 of Chapter III, introduces the notion of domination between p^{th} order processes. A necessary condition for equivalence of two Gaussian processes, namely the setwise equality of their reproducing kernel Hilbert spaces, is shown to be true for symmetric stable processes with the function space introduced in Chapter II replacing the reproducing kernel Hilbert space. Further, for p^{th} order processes with $1 < p < 2$, necessary conditions for absolute contintuity and sufficient conditions for singularity are presented analogous to those of Fortet (1973) for second order processes.

Finally, Section 3 of Chapter III makes use of the results of Section 3.1 to show that a dichotomy holds for certain symmetric stable processes including independently scattered random measures and harmonizable processes. Necessary and sufficient conditions for equivalence and singularity are given. The singularity between an invertible symmetric stable process and its multiples is also proved.

1.3 Background and notation

The following setting is considered. $X = (X(t) = X(t,\omega); t \in T)$ is a stochastic process on a probability space $(\Omega, \mathcal{F}, P)$ with parameter set T and real or complex values, i.e. values in $\mathbf{X} = \mathbb{R}$ or $\mathbb{C}$. When $X(t) \in L_p(\Omega,\mathcal{F},P) = L_p(P)$ for all $t \in T$, and some $p > 0$, X is called a pth order process. The iinear space $\mathcal{L}(X)$ of a pth order process X is the $L_p(P)$ completion of the set of finite linear combinations of its random variables $l(X) \triangleq sp\{X(t); t \in T\}$. $\overline{\mathbf{X}}^T$ denotes the set of all extended $\mathbf{X}$-valued (i.e.. real or complex valued) functions on T, $\mathcal{C} = \mathcal{C}(\overline{\mathbf{X}}^T)$ the σ-field generated by the cylinder sets of $\overline{\mathbf{X}}^T$ and μ_X the distribution of the process X. i.e. the probability induced on $\mathcal{C}$ by X:

$$\mu_X (C) = P(\{\omega; X(\cdot,\omega) \in C\}), \ C \in \mathcal{C}.$$

For two stochastic processes X and Y we are interested in the Lebesgue

decomposition of the distribution μ_Y of Y with respect to the distribution μ_X of X, and in particular in conditions for μ_Y and μ_X to be singular ($\mu_Y \perp \mu_X$), and for μ_Y to be absolutely continuous with respect μ_X ($\mu_Y \ll \mu_X$). If the two measures μ_Y and μ_X are mutually absolutely continuous we say that they are equivalent ($\mu_Y \sim \mu_X$). Of particular interest is the case where $Y = s + X$ for a nonrandom function s on T. The function s is then called a singular or admissible translate of X if $\mu_{s+X} \perp \mu_X$ or $\mu_{s+X} \ll \mu_X$ respectively.

Here we focus primarily on symmetric α-stable (SαS) processes. A real random variable X is SαS, $0 < \alpha \le 2$, with scale parameter $\|X\|_\alpha \in (0,\infty)$ if $E\{\exp(iuX)\} = \exp\{-\|X\|_\alpha^\alpha |u|^\alpha\}$. A real random vector $(X_1,...,X_n)$ is SαS (or its components are jointly SαS) if all linear combinations $\sum_{k=1}^n a_k X_k$ are SαS. Similarly a real stochastic process $X = (X(t); t \in T)$ is SαS if all linear combinations $\sum_{k=1}^n a_n X(t_k)$ are SαS random variables. When $\alpha = 2$ we have zero mean Gaussian random variables, vectors and processes respectively. When $0 < \alpha < 2$, the tails of the distributions are heavier and only moments of order $p \in (0,\alpha)$ are finite with

$$\{E(|X|^p)\}^{1/p} = C_{p,\alpha}\|X\|_\alpha,$$

where the constant $C_{p,\alpha}$ is independent of X. Thus a SαS process X is p^{th} order for all $0 < p < \alpha$, and its linear space $\mathcal{L}(X)$ does not depend on p and is the completion of $l(X)$ with respect to $\|\cdot\|_\alpha^{1 \wedge \alpha}$, which in fact metrizes convergence in probability (Schilder (1970)).

An important class of SαS processes consists of SαS independently scattered random measures, which extend the concept of a stochastic process with independent increments to more general parameter spaces. Let I be an arbitrary set and $\mathfrak{I}$ a δ-ring of subsets of I with the property that there exists an increasing sequence $(I_n; n \in \mathbb{N})$ in $\mathfrak{I}$ with $\cup_n I_n = I$. A real stochastic process $Z = (Z(A); A \in \mathfrak{I})$ is called an independently

scattered SαS random measure if for every sequence $(A_n; n \in \mathbf{N})$ of disjoint sets in $\mathfrak{I}$, the random variables $\{Z(A_n); n \in \mathbf{N}\}$ are independent, and whenever $\cup_n A_n \in \mathfrak{I}$ then $Z(\cup_n A_n) = \sum_n Z(A_n)$ a.s., and for every $A \in \mathfrak{I}$, $Z(A)$ is a SαS random variable, i.e. $E\{\exp(iuZ(A))\} = \exp\{-m(A)|u|^{\alpha}\}$ where $m(A) = \|Z(A)\|_{\alpha}^{\alpha}$. Then m is a measure on $\mathfrak{I}$ which extends uniquely to a σ-finite measure on $\sigma(\mathfrak{I})$, and is called the control measures of Z. Conversely, the existence of an independently scattered SαS random measure with a given control measure is a consequence of Komogorov's consistency theorem.

When **I** is an interval of the real line, there is an identification between independent increments processes and independently scattered random measures. Namely if $X = (X(t), t \in \mathbf{I})$ is an independent increments process and $(a,b) \subset \mathbf{I}$: an interval, $Z((a,b]) \triangleq X(b) - X(a)$ can be extended to an independently scattered random measure on the δ-ring $\mathfrak{I}$ of bounded Borel sets of **I**. Conversely given an independently scattered random measure Z on $\mathfrak{I}$, and a in **I**, $X(t) = \text{sign}(t-a)Z((a \wedge t, a \vee t])$, $t \in \mathbf{I}$, is an independent increments process. When the control measure m is Lebesgue measure, then X has stationary independent increments, $E\{\exp(iu[X(t) - X(t')])\} = \exp\{-|t - t'| \, |u|^{\alpha}\}$, and is called S$\alpha$S motion on **I**.

For any function $f \in L_{\alpha}(\mathbf{I}, \sigma(\mathfrak{I}), m) = L_{\alpha}(m)$ the stochastic integral $\int_{\mathbf{I}} f dZ$ can be defined in the usual way and is a SαS random variable with $\|\int_{\mathbf{I}} f dZ\|_{\alpha} = \|f\|_{L_{\alpha}(m)}$. The stochastic integral map $f \rightarrow \int_{\mathbf{I}} f dZ$ from $L_{\alpha}(m)$ into $\mathcal{L}(Z)$ is an isometry and

$$(1.3.1) \qquad \mathcal{L}(Z) = \{\textstyle\int f dZ; f \in L_{\alpha}(m)\}.$$

The stochastic integral allows for the construction of SαS processes with generally dependent values by means of the spectral representation

$$(1.3.2) \qquad X(t) = \int_{\mathbf{I}} f(t,u) Z(du), \quad t \in \mathbf{T},$$

where $\{f(t,\cdot); t \in T\} \subset L_\alpha(m)$. In fact every S$\alpha$S process X has such a spectral

representation in law, in the sense that for some family $\{f(t, \cdot), t \in T\}$ in some $L_\alpha(m)$,

$$(1.3.3) \qquad (X(t); t \in T) \overset{L}{=} (\int_I f(t,u)Z(du); t \in T)$$

(see e.g., Kuelbs (1973) and Hardin (1982)). If $\mathcal{L}(X)$ is separable, e.g. X is continuous in

probability, then $L_\alpha(I,m)$ can be chosen as $L_\alpha([0,1],\text{Leb})$. Specific examples of S$\alpha$S

processes will be considered in the following sections.

The covariation $[X,Y]_\alpha$ of two jointly SαS random variables X and Y with

$1 < \alpha \le 2$ is defined by

$$(1.3.4) \qquad \frac{[X,Y]_\alpha}{\|Y\|_\alpha^\alpha} = \frac{E(XY^{<p-1>})}{E(|Y|^p)},$$

which holds for all $0 < p < \alpha$, where $y^{<q>} = |y|^{q-1}y$, $q > 0$ (see e.g. Cambanis and

Miamee (1985)). It follows that $\|X\|_\alpha^\alpha = [X,X]_\alpha$. If X and Y have representations

$\int_I f dZ$ and $\int_I g dZ$ respectively then $[X,Y]_\alpha = \int_I f g^{<\alpha-1>} dm$.

In certain cases, such as when working with Fourier transforms, it is more natural

and convenient to work with complex valued processes. A complex SαS random variable

is defined as having jointly SαS real and imaginary parts. Except for the representation

of the characteristic function, all concepts and results considered in this section for real

SαS random variables and processes extend to the complex case (see e.g. Cambanis

(1982) and Cambanis and Miamee (1985)).

CHAPTER II

ADMISSIBLE AND SINGULAR TRANSLATES

2.1 <u>An upper bound for the set of admissible translates</u>

A space of functions associated with a pth order, $0 < p \leq 2$, stochastic process will be introduced and seen as a partial extension of the reproducing kernel Hilbert space (RKHS) associated with a second order process. We concentrate only on p^{th} order processes with $p < 2$ because for those with $p \geq 2$ the second order theory is applicable.

Recall that for a second order stochastic process $X = (X(t); t \in T)$ with arbitrary index set T, zero mean and covariance function R, the RKHS H of X (or of R) consists of all functions s of the form $s(t) = E(X(t)\overline{Y})$, $t \in T$, $Y \in \mathcal{L}(X)$. If $s_i(t) = E(X(t)\overline{Y}_i)$ then $<s_1,s_2>_H = E(Y_1\overline{Y}_2)$ defines an inner product and R is a reproducing kernel, i.e. for all $t \in T$, $R(\cdot,t) \in H$ and $s(t) = <s,R(\cdot,t)>_H$. Also $s \in H$ if and only if

$$\|s\|_H = \sup\frac{|\sum_{n=1}^{N}a_n s(t_n)|}{[E|\sum_{n=1}^{N}a_n X(t_n)|^2]^{1/2}} < \infty,$$

where the supremum is taken over all $N \in \mathbb{N}$, $a_1,...,a_N \in X$ and $t_1,...,t_N \in T$ (see e.g. Fortet (1973)).

We now introduce the function space of a pth order process with $0 < p \leq 2$ and arbitrary index T, and present its properties.

<u>Definition 2.2.1</u>. The function space $F = F(X)$ of a pth order process $X = (X(t); t \in T)$ with $0 < p \leq 2$ is the set of all functions s on T such that

$$\|s\|_{\mathbf{F}} \triangleq \sup \frac{|\sum_{n=1}^{N} a_n s(t_n)|}{[E|\sum_{n=1}^{N} a_n X(t_n)|^p]^{1/p}} < \infty,$$

where the supremum is taken over all $N \in \mathbf{N}$, $a_1,...,a_N \in \mathbf{X}$ and $t_1,...,t_N \in \mathbf{T}$.

When $1 < p \leq 2$, a representation is known for the bounded linear functionals on the linear space of X, analogous to the Riesz representation for bounded linear functionals on a Hilbert space. This allows us to express the functions in $\mathbf{F}$ in terms of moments of the process X. This and further properties of the function space are collected in the following

<u>Proposition 2.1.2</u>. Let $X = (X(t); t \in \mathbf{T})$ be a pth order process with $1 < p \leq 2$. Then the following three statements are equivalent:

    i) $s \in \mathbf{F}$,

    ii) $s(t) = E(X(t)Y^{<p-1>})$ for $Y \in \mathcal{L}(X)$,

    iii) $s(t) = E(X(t)\overline{W})$ for $W \in L_{p^*}(P)$ where $1/p + 1/p* = 1$.

Moreover the following properties hold.

    a) $\|s\|_{\mathbf{F}} = \|Y\|_{L_p(P)}^{p-1}$ if $s(t) = E(X(t)Y^{<p-1>})$, $Y \in \mathcal{L}(X)$.

    b)   For each $s \in \mathbf{F}$, with $s(t) = E(X(t)Y^{<p-1>})$, $Y \in \mathcal{L}(X)$, there exists a unique $W \in L_{p^*}(P)$ (namely $\overline{W} = Y^{<p-1>}$) satisfying iii) and $\|s\|_{\mathbf{F}} = \|W\|_{L_{p^*}(P)}$.

    c) $(\mathbf{F}, \|\cdot\|_{\mathbf{F}})$ is a Banach space isometrically isomorphic to the quotient space $L_{p*}(P)/l(X)^{\perp}$, where $l(X)^{\perp}$ denotes the annihilator of $l(X)$.

<u>Proof:</u>   i) $\Rightarrow$ ii) follows by observing that if $\|s\|_{\mathbf{F}} < \infty$, then $\psi_s(\sum_{n=1}^{N} a_n X(t_n)) = \sum_{n=1}^{N} a_n s(t_n)$ defines a bounded linear functional on $\mathcal{L}(X)$ with norm $\|s\|_{\mathbf{F}}$. From Cambanis and Miller (1981), Proposition 2.1, there exists a unique $Y \in \mathcal{L}(X)$ such that $\psi_s(\cdot) = E(\cdot Y^{<p-1>})$ and $\|\psi_s\|_{\mathcal{L}(X)^*} = \|Y\|_{L_p(P)}^{p-1}$. Thus $s(t) = \psi_s(X(t)) = E(X(t)Y^{<p-1>})$.

ii) $\Rightarrow$ iii) If $Y \in \mathcal{L}(X)$ then $\overline{W} = Y^{<p-1>} \in L_{p*}(P)$ and $\|W\|_{L_{p*}(P)} = \|Y\|_{L_p(P)}^{p-1}$. Also $s(t) = E(X(t)Y^{<p-1>}) = E(X(t)\overline{W})$.

iii) $\Rightarrow$ i) If $s(t) = E(X(t)\overline{W})$ then it is clear from its definition that $\|s\|_{\mathbf{F}}$ is finite.

a) That $\|s\|_{\mathbf{F}} = \|Y\|_{L_p(P)}^{p-1}$ follows as in the proof of i) $\Rightarrow$ ii).

b) Let $s \in \mathbf{F}$. By iii) there exist $Z \in L_{p*}(P)$ such that $s(t) = E(X(t)\overline{Z})$. Let $l(X)^{\perp}$ be the closed linear space

$$\{Z' \in L_{p*}(\Omega); E(Z'\overline{Y}) = 0, Y \in l(X)\},$$

and let Z_0 be the best approximation of X in $l(X)^{\perp}$ i.e.

$$\|Z-Z_0\|_{L_{p*}(P)} = \inf\{\|Z-Z'\|_{L_{p*}(P)}; Z' \in l(X)^{\perp}\}.$$

Such a $Z_0 \in l(X)^{\perp}$ exists and is unique (Singer (1970), Corollary 3.5 and Theorem 1.11). Set $W = Z-Z_0$. Then $E(Z\overline{Y}) = E(W\overline{Y})$ for all $Y \in l(X)$. If Z' is such that $E(Z'\overline{Y}) = E(Z\overline{Y})$ for all $Y \in l(X)$, then $Z-Z' \in l(X)^{\perp}$ and

$$\|W\|_{L_{p*}(P)} = \|Z-Z_0\|_{L_{p*}(P)} \leq \|Z-(Z-Z')\|_{L_{p*}(P)} = \|Z'\|_{L_{p*}(P)}.$$

Thus if $s(t) = E(X(t)\overline{W}')$, $W' \in L_{p*}(P)$, and $\|s\|_{\mathbf{F}} = \|W'\|_{L_{p*}(P)}$ we must have $\|W\|_{L_{p*}(P)} \leq \|W'\|_{L_{p*}(P)}$. On the other hand

$$\|W'\|_{L_{p*}(P)} = \|s\|_{\mathbf{F}} = \sup \frac{|E(\sum_{n=1}^{N} a_n X(t_n)W)|}{\|\sum_{n=1}^{N} a_n X(t_n)\|_{L_p(P)}} \leq \|W\|_{L_{p*}(P)}.$$

Therefore $\|W\|_{L_{p*}(P)} = \|W'\|_{L_{p*}(P)} = \|s\|_{\mathbf{F}}$. Putting $V = Z-W'$ we have $V \in l(X)^{\perp}$ and

$$\|Z-Z_0\|_{L_{p*}(P)} = \|W\|_{L_{p*}(P)} = \|W'\|_{L_{p*}(P)} = \|Z-V\|_{L_{p*}(P)}.$$

Thus the unicity of Z_0 implies $W = W'$. Since $Y^{<p-1>} \in L_{p*}(P)$ and
$\|s\|_F = \|Y\|_{L_{p*}(P)}^{<p-1>}$ for $s(t) = E(X(t)Y^{<p-1>})$ we must have $W = Y^{<p-1>}$.

    c)   That $(F, \|\cdot\|_F)$ is a normed linear space is clear. To show that F is
isometrically isomorphic to $L_{p*}(P)/l(X)^\perp$, let $s_i \in F, i = 1,2,$

$$s_i(t) = E(X(t)\overline{W}_i), \quad \text{and} \quad (s_1+s_2)(t) = E(X(t)\overline{W}),$$

where W_1, W_2 and W are the unique elements in $L_{p*}(P)$ such that

$$\|s_i\|_F = \|W_i\|_{L_{p*}(P)}, \quad \text{and } \|s_1+s_2\|_F = \|W\|_{L_{p*}(P)}.$$

Since

$$E(X(t)W) = (s_1+s_2)(t) = E(X(t)(W_1+W_2))$$

we have $W - (W_1+W_2) \in l(X)^\perp$ i.e. $[W] = [W_1+W_2] = [W_1]+[W_2]$, where $[\,\cdot\,]$
denotes an equivalence class in $L_{p*}(P)/l(X)^\perp$. Similarly if $s(t) = E(X(t)\overline{W})$ and
$(as)(t) = E(X(t)\tilde{W})$ we have $[\tilde{W}] = [aW] = a[W]$. Hence the map $s \to [W]$ is linear
and since

$$\|[W]\|_{L_{p*}(P)/l(X)^\perp} = \|W\|_{L_{p*}(P)} = \|s\|_F$$

it is an isometric isomorphism.

    To finish the proof of c) we need to show that F is complete. Let $(s_k; k \in N)$ be a
sequence in F such that $\sum_{k=1}^\infty \|s_k\|_F < \infty$ and let $W_k \in L_{p*}(P)$ be such that
$\|W_k\|_{L_{p*}(P)} = \|s_k\|_F$. Hence $\sum_{k=1}^\infty \|W_k\|_{L_{p*}(P)} < \infty$ and $W = \sum_{k=1}^\infty W_k \in L_{p*}(P)$.

Set $s(t) = E(X(t)\overline{W})$. Thus

$$|\sum_{n=1}^{N} a_n (\sum_{k=1}^{K} s_k - s)(t_n)| \leq \|\sum_{k=1}^{K} W_k - W\|_{L_{p*}(P)} \|\sum_{n=1}^{N} a_n X(t_n)\|_{L_p(P)}$$

and

$$\|\sum_{k=1}^{K} s_k - s\|_F \leq \|\sum_{k=1}^{K} W_k - W\|_{L_{p*}(P)} \to 0 \text{ as } K \to \infty,$$

i.e. $\sum_{k=1}^{\infty} s_n \in F$ proving that F is complete. □

Further properties of the function space F of the process X, for $1 < p \leq 2$, analogous to those of a RKHS are the following:

i)   If T is a metric space, functions in F are as "smooth" as the process X is in the weak sense, i.e., they are continuous (differentiable) if and only if X is weakly continuous (differentiable).

ii)   Norm convergence in F implies pointwise convergency, and the convergence is uniform if $\|X(t)\|_{L_p(P)}$ is uniformly bounded.

If the process X is SαS with $1 < \alpha < 2$, then it is of pth order for each $p \in (1,\alpha)$ and its function space F does not depend on p but only on α and can be defined by means of moments. Furthermore the functions in F can be expressed in terms of the spectral representation of the process.

Corollary 2.1.3. Let $X = (X(t); t \in T)$ be a SαS process with $1 < \alpha < 2$ and spectral representation

$$X(t) = \int_I f(t,u) Z(du), \ t \in T,$$

where Z has control measure m. Then the following three statements are equivalent

i) $s \in F$,

ii) $s(t) = [X(t),Y]_\alpha$ for $Y \in \mathcal{L}(X)$,

iii) $s(t) = \int_I f(t,u)\bar{z}(u)m(du)$ for $z \in L_{\alpha^*}(m)$ where $1/\alpha^* + 1/\alpha = 1$.

Moreover the following properties hold.

a) $\quad \|s\|_{\mathbf{F}} = C_{p,\alpha} \sup \dfrac{|\sum_{n=1}^{N} a_n s(t_n)|}{\|\sum_{n=1}^{N} a_n X(t_n)\|_\alpha} = \|Y\|_\alpha^{\alpha-1}.$

b) $\quad$ For each $s \in \mathbf{F}$ there exists a unique $z \in L_{\alpha^*}(m)$ satisfying iii) and $\|s\|_{\mathbf{F}} = \|z\|_{L_{\alpha^*}(m)}.$

c) $\quad$ The map $s \to [z]$ from $\mathbf{F}$ into $L_{\alpha^*}(m)/l(f)^{\perp}$, where $[\ \cdot\ ]$ is an equivalence class in $L_{\alpha^*}(m)/l(f)^{\perp}$, $l(f) = sp\{f(t,\cdot); t \in \mathbf{T}\}$, is an isometric isomorphism.

<u>Proof</u>: $\quad$ i) $\Leftrightarrow$ ii). It follows from 1.3.3 that for all $p \in (1,\alpha)$,

$$[\cdot, Y]_\alpha = E(\ \cdot\ Z^{<p-1>}).$$

where

$$Z = C_{p,\alpha}^{-p/(p-1)} \|Y\|_\alpha^{(\alpha-p)/(p-1)} Y \ \text{ and } \ \|Z\|_{L_p(P)}^{p-1} = C_{p,\alpha}^{-1}\|Y\|_\alpha^{\alpha-1},$$

so that $s \in \mathbf{F}$ if and only if $s(t) = [X(t),Y]_\alpha$ which does not depend on p.

ii) $\Leftrightarrow$ iii). If $Y \in \mathcal{L}(X)$ then $Y = \int_{\mathbf{I}} g dZ$ for some $g \in \mathcal{L}(f) = \overline{sp}\{f(t,\cdot); t \in \mathbf{T}\}$, and

$$s(t) \ = \ [X(t),Y]_\alpha = [\textstyle\int_{\mathbf{I}} f(t)dZ, \int_{\mathbf{I}} g dZ]_\alpha \ =$$

$$= \ \textstyle\int_{\mathbf{I}} f(t)g^{<\alpha-1>} dm \ = \ \int_{\mathbf{I}} f(t)z dm,$$

where $z = g^{<\alpha-1>} \in L_{\alpha*}(m)$ and $\|z\|_{L_{\alpha*}(m)} = \|Y\|_\alpha^{\alpha-1}.$

The proofs of iii) $\Rightarrow$ i), the uniqueness of z and of the isometric isomorphism are identical to those of Proposition 2.1. $\qquad\qquad \square$

Further, the well known dichotomy on the admissible translates of a Gaussian process — namely that the admissible translates of a Gaussian process are precisely the functions in its RKHS, and its translates by functions outisde its RKHS are singular — has a partial analog for p^{th} order processes $0 < p \leq 2$, where the RKHS is replaced by the function space $\mathbf{F}$. Our result extends that of Théorème 4.1 in Fortet (1973) to p^{th} order processes with $0 < p < 2$, and when applied to SαS processes with $0 < \alpha < 2$, it generalizes Proposition 10 in Zinn (1975) to any SαS process.

<u>Proposition</u> 2.1.4. Let $X = (X(t); t \in \mathbf{T})$ be a pth order process with $0 < p \leq 2$. If $s \notin \mathbf{F}$ then $\mu_{s+X} \perp \mu_X$. Consequently all admissible translates of X belong to $\mathbf{F}$.

<u>Proof</u>. The proof is adapted from Pang (1973). If $s \in \mathbf{F}$, then

$$\sup \frac{|\sum_{n=1}^{N} a_n s(t_n)|}{\|\sum_{n=1}^{N} a_n X(t_n)\|_{L_p(P)}} = \infty.$$

Hence for each $n \in \mathbf{N}$, we can choose $N_n, a_{n,k}, t_{n,k}, k = 1,...,N_n$ such that

$$\frac{|\sum_{k=1}^{N_n} a_{n,k} s(t_{n,k})|}{\|\sum_{k=1}^{N_n} a_{n,k} X(t_{n,k})\|_{L_p(P)}} \geq n^{1/p}.$$

Let $s_n = \sum_{k=1}^{N_n} a_{n,k} s(t_{n,k})$. Without loss of generality we can consider $s_n > 0$ for all n. Consider the random variables defined on $(\mathbf{X}^{\mathbf{T}}, \mathcal{C}, \mu_X)$ by

$$Y_n(x) = \sum_{k=1}^{N_n} a_{n,k} x(t_{n,k}), \quad x \in \mathbf{X}^{\mathbf{T}}.$$

By the Markov inequality we have

$$\mu_X(Y_n \geq s_n/2) \leq \mu_X(|Y_n| \geq s_n/2)$$

$$\leq 2^P \|Y_n\|^p_{L_p(P)}/s_n^p < 2^P/n \to 0 \qquad \text{as } n \to \infty$$

and

$$
\begin{aligned}
\mu_{s+X}(Y_n \geq s_n/2) \quad &= \mu_X(Y_n + s_n \geq s_n/2) \\
&= \mu_X(Y_n \geq -s_n/2) \\
&\geq \mu_X(|Y_n| < s_n/2) \\
&= 1 - \mu_X(|Y_n| \geq s_n/2) \\
&\geq 1 - 2^P \|Y_n\|^p_{L_p(P)}/s_n^p \\
&\geq 1 - 2^P/n \to 1 \qquad \text{as } n \to \infty.
\end{aligned}
$$

Therefore $\mu_X \perp \mu_{s+X}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Restricting our attention to SαS processes we see that in contrast with the Gaussian case, $\alpha = 2$, where the set of admissible translates is always the entire space F, i.e. the RKHS, the set of admissible translates of a SαS process with $\alpha < 2$ may be as large as the entire function space F or as small as $\{0\}$, as is seen by the following examples.

<u>Stable Motion</u>: If $X = (X(t); t \in [0,1])$ is a SαS motion, i.e., X has stationary independent SαS increments, it is known (Brockett and Tucker (1977), Gihman and Skorohod (1966), Zinn (1975)), that X has no nontrivial admissible translates for $0 < \alpha < 2$. On the other hand for $1 < \alpha < 2$, its function space is the space of absolutely continuous functions with $s(0) = 0$ and derivative in $L_{\alpha^*}(\text{Leb})$, i.e.

$$F = \{s;\ s(t) = \int_0^t s'(u)du,\ t \in [0,1],\ s' \in L_{\alpha^*}(\text{Leb})\}$$

with $\|s\|_F = \|s'\|_{L_{\alpha^*}(\text{Leb})}$.

<u>Sub-Gaussian processes</u>: Let $X = (X(t); t \in \mathbf{T})$ be an α-sub-Gaussian process, i.e. its finite dimensional characteristic functions have the form

$$E\{\exp(i\textstyle\sum_{n=1}^{N} a_n X(t_n))\} = \exp\{-(\tfrac{1}{2}\textstyle\sum_{n,m=1}^{N} a_n R(t_n,t_m) a_m)^{\alpha/2}\},$$

where R is a covariance function, or equivalently

$$(X(t); t \in \mathbf{T}) \overset{\mathrm{L}}{=} (A^{1/2} G(t); t \in \mathbf{T}),$$

where A is a normalized positive $(\alpha/2)$-stable random variable independent of the Gaussian process $G = (G(t); t \in \mathbf{T})$ which has zero mean and covariance function R. It follows from Huang and Cambanis (1979) that the set of admissible translates of X coincides with the RKHS of G, once we observe that there the proof depends only on the representation of spherically invariant processes as scale mixtures of Gaussian processes and not on the existence of second moments. Moreover for any $Y \in \mathcal{L}(X)$,

$$[X(t),Y]_\alpha = 2^{\alpha/2}\{E(W^2)\}^{1-\alpha/2} E(G(t)W),$$

where $W \in \mathcal{L}(G)$ is obtained from G by the same linear operation Y is obtained from X (see Cambanis and Miller (1981)). Therefore the function space $\mathbf{F}$ of X coincides with the RKHS of G and is therefore a Hilbert space.

<u>Stable processes as mixtures of Gaussian processes</u>. It has been shown in LePage (1980) that every SαS process X is conditionally Gaussian with zero mean, i.e. there exists a sub-σ-field $\mathcal{G}$ of $\mathcal{F}$ such that given $\mathcal{G}$, the law of X is Gaussian with mean zero and covariance function R. Denoting by G_R such a Gaussian process and by μ_{G_R} its distribution, we have that for every SαS process X there exists a probability λ on the

space $\Re$ of all covariance functions R such that

$$\mu_X(E) = \int_{\Re} \mu_{G_R}(E)\lambda(dR)$$

for all $E \in \mathcal{C}$. The SαS process X is thus a Gaussian process with random covariance function R, and it is easily checked that all quadratic forms $\sum_{n,m=1}^{N} a_n R(t_n, t_m)\bar{a}_m$ are positive $(\alpha/2)$-stable random variables. Likewise we have for all $E \in \mathcal{C}$,

$$\mu_{s+X}(E) = \int_{\Re} \mu_{s+G_R}(E)\lambda(dR).$$

It follows that if s is an admissible translate of almost all G_R's, then it is an admissible translate of X too. This gives a lower bound for the set of admissible translates of X, namely

$$\bigcup_{\substack{\Lambda \subset \Re \\ \lambda(\Lambda)=0}} \bigcap_{R \in \Re \setminus \Lambda} \text{RKHS}(R).$$

Thus a SαS process will have admissible translates if it is a mixture of Gaussian processes whose RKHS's have a common part, i.e. if $\bigcap_{R \in \Re \setminus \Lambda} \text{RKHS}(R) \neq \{0\}$ for some $\lambda(\Lambda) = 0$.

The converse does not seem to be necessarily true, i.e. an admissible translate of X may not be an admissible translate of almost all the Gaussian processes whose mixture is X.

It also follows that a singular translate of X is a singular translate of almost all the Gaussian processes whose mixture is X, and furthermore the same event separates them. This gives an upper bound for the set of singular translates of X, namely

$$\bigcup_{\substack{\Lambda \subset \Re \\ \lambda(\Lambda)=0}} \bigcap_{R \in \Re \setminus \Lambda} RKHS(R)^c.$$

Conversely, if s is a singular translate of a.e. $G_R(\lambda)$, it may not be a singular translate of X; but if furthermore the separating set of μ_{s+G_R} and μ_{G_R} does not depend on R a.e. (λ), then s is a singular translate of X.

When a SαS process is a mixture of Gaussian processes having the same RKHS then we show that a dichotomy prevails, with every translate being either admissible or singular.

<u>Proposition</u> 2.1.5. Let the SαS process $X = (X(t); t \in \mathbf{T})$ be the λ-mixture of Gaussian processes $G_R = (G_R(t); t \in \mathbf{T})$ such that $RKHS(R) = H$ a.e. (λ). Then s is an admissible translate of X if and only if $s \in H$, and s is a singular translate of X if and only if $s \notin H$.

<u>Proof</u>: If $s \in H$, then s is an admissible translate of a.e. $G_R(\lambda)$, and hence of X.

Now assume $s \notin H$. Let $RKHS(R) = H$ for all $R \in \Re \setminus \Lambda$, $\lambda(\Lambda) = 0$, and fix $R_0 \in \Re \setminus \Lambda$. Then for each $n \in \mathbf{N}$, there exit N_n, $a_{n,1},\dots,a_{n,N_n}$, $t_{n,1}, \dots, t_{n,N_n}$ such that

$$\frac{|\sum_{k=1}^{N_n} a_n s(t_{n,k})|^2}{E|\sum_{k=1}^{N_n} a_{n,k} G_{R_0}(t_{n,k})|^2} \geq n.$$

Since for every $R \in \Re \setminus \Lambda$, $RKHS(R) = H$, there exists $0 < c_R < \infty$ such that

$$E|\sum_{k=1}^{N_n} a_{n,k} G_R(t_{n,k})|^2 \leq c_R E|\sum_{k=1}^{N_n} a_{n,k} G_{R_0}(t_{n,k})|^2.$$

As in Proposition 2.1.4, let $s_n = \sum_{k=1}^{N_n} a_{n,k} s(t_{n,k})$. (and WLOG assume $s_n > 0$) and $Y_n(x) = \sum_{k=1}^{N_n} a_{n,k} x(t_{n,k})$, $x \in \mathbf{X}^{\mathbf{T}}$, so that

$$\mu_{G_R}(Y_n \geq s_n/2) \quad \leq \quad 2^2 E|\sum_{k=1}^{N_n} a_{n,k} G_R(t_n)|^2/s_n^2$$

$$\leq \quad 4c_R E|\sum_{k=1}^{N_n} a_{n,k} G_{R_0}(t_n)|^2/s_n^2$$

$$\leq \quad 4c_R/n \to 0 \qquad \text{as } n \to \infty,$$

and

$$\mu_{s+G_R}(Y_n \geq s_n/2) \quad \geq 1 - \mu_{G_R}(|Y_n| > s_n/2)$$

$$\geq 1 - 4c_R/n \to 1 \qquad \text{as } n \to \infty.$$

Hence by the dominated convergence theorem

$$\mu_X(Y_n \geq s_n/2) \;=\; \int_{\Re} \mu_{G_R}(Y_n \geq s_n/2)\lambda(dR) \;\to\; 0 \qquad \text{as } n \to \infty,$$

and

$$\mu_{s+X}(Y_n \geq s_n/2) \;=\; \int_{\Re} \mu_{s+G_R}(Y_n \geq s_n/2)\lambda(dR) \;\to\; 1 \qquad \text{as } n \to \infty.$$

This implies $\mu_X \perp \mu_{s+X}$. Hence every $s \notin H$ is a singular translate of X, and the proof of the dichotomy is complete. $\qquad\qquad \square$

The assumptions of Proposition 2.1.5 are satisfied when X is sub-Gaussian, i.e. X is the mixture of the mutually singular Gaussian process $a^{1/2}G$, $a > 0$, which have identical RKHS; or in the more general case where X is the mixture of Gaussian processes with random covariance function of the form $\sum_{n=1}^{N} A_n R_n(t,s)$, where the R_n's are fixed (nonrandom) covariance such that $R_n - c_{nm}R_m$ is nonnegative definite for all $n,m = 1,...,N$, $n \neq m$, and some $0 < c_{nm} < \infty$, and the positive random variables $A_1,...,A_n$, are jointly $(\alpha/2)$-stable.

The usefulness of these general remarks is limited by the fact that the only $S\alpha S$ mixtures of Gaussian processes, which are currently known explicitly, are the sub-Gaussian processes, and the more general finite sums $\sum_{n=1}^{N} A_n^{1/2} G_n$, where $(A_1,...,A_N)$

is positive $(\alpha/2)$-stable and independent of the mutually independent Gaussian processes $G_1,...,G_N$.

Further examples where the set of admissible translates is trivial or a proper subset of the function space F are presented in the next section. It should finally be recalled that the set of admissible translates of a SαS process is always a linear space, even if it is not the entire function space F (Zinn (1975), Corollary 5.1). However, as will be seen in the next section, the restriction of $\|\cdot\|_F$ to the set of admissible translates may not be the most natural way to define a topology on it. Also, from the linear structure we have that $\mu_{s+X} \ll \mu_X \Rightarrow \mu_X \ll \mu_{s+X}$ (see e.g. Thang and Tien (1979)) so that for every admissible translate s, μ_{s+X} and μ_X are equivalent.

2.2 Translates of invertible processes

In this section we present some general results on the admissible translates of certain SαS processes with invertible spectral representation.

Let $X = (X(t); t \in T)$ be a SαS stochastic process with spectral representation as in (1.3.2). It follows from the continuity of the stochastic integral map $f \to \int f dZ$ and (1.3.1) that the representing functions $\{(f(t,\cdot); t \in T)\}$ are linearly dense in $L_\alpha(m)$, i.e. that $\pounds(f) = L_\alpha(m)$, where $\pounds(f)$ is the completion of $\ell(f) = \mathrm{sp}\{(f(t,\cdot); t \in T)\}$ in $L_\alpha(m)$, if and only if $\pounds(X) = \pounds(Z)$. Processes satisfying this condition will be said to have an invertible spectral representation or more simply to be invertible.

Every Gaussian process is invertible Cambanis (1975), construction in Theorem 2. This is not generally true for non-Gaussian SαS processes as can be seen from the fact that the linear space of a sub-Gaussian process does not contain (nontrivial) independent random variables (Cambanis and Soltani (1982), Lemma 2.1). Necessary and sufficient conditions for a general SαS process to have an invertible spectral representation are given in Cambanis (1982), Theorems 5.1 and 5.5. A stronger form of invertibility for a nonanticipating SαS moving average is considered in Cambanis and Soltani (1982).

Lemma 3.1. SαS processes with invertible spectral representation in $L_2([0,1], \text{Leb})$, i.e. $L_2\text{-}\overline{\text{sp}}\{f(t,\cdot);\ t \in [0,1]\} = L_2([0,1], \text{Leb})$, are considered in Zinn (1975); clearly such a process has also invertible spectral representation in $L_\alpha([0,1], \text{Leb})$. Examples of invertible SαS processes will be presented in the sequel.

For invertible processes the problem of finding their admissible translates can be reduced to finding the admissible translates of the independently scattered random measure Z, which we now consider first.

The next proposition is essentially based on Gihman and Skorohod (1966), Theorem 7.3. It extends to independently scattered SαS random measures with non-atomic control measure the result in Brockett and Tucker (1977) and Zinn (1975) on admissible translates of independent increments processes in $[0,T]$ which are stochastically continuous and have no Gaussian component. It establishes a dichotomy for the translates of a general independently scattered SαS random measure and it characterizes its admissible translates as those of its atomic component.

The following notation will be used in Proposition 2.2.1. Recall that if a σ-finite measure space $(\mathbf{l}, \sigma(\mathfrak{I}), m)$ is such that $\sigma(\mathfrak{I})$ contains all single points sets (e.g. $\mathbf{l}$ is a Polish space, $\sigma(\mathfrak{I})$ its Borel sets, and $\mathfrak{I}$ the δ-ring of Borel sets with finite m-measure) then $m = m_a + m_d$ where m_a is purely atomic and m_d is diffuse (non-atomic) (Kingman and Taylor (1966)), and the set of atoms is at most countable, say $A = \{a_n;\ n \in \{1,2,...,N\} \cap \mathbf{N}\}$, N the number of atoms. Thus if $Z = (Z(B):\ B \in \mathfrak{I})$ is an independently scattered SαS random measure with control measure m, it can be expressed as

$$Z = Z_a + Z_d,$$

where Z_a and Z_d are independent SαS independently scattered random measures defined for all $B \in \mathfrak{I}$ by

$$Z_a(B) = Z(A \cap B) \quad \text{and} \quad Z_d(B) = Z(A^c \cap B),$$

and have control measures m_a and m_d respectively. The atomic component has a series expansion

$$Z_a(B) = \sum_{n=1}^{N} 1_B(a_n) Z(\{a_n\})$$

which can be normalized by using the i.i.d. standard $S\alpha S$ random variables

$$Z_n \triangleq Z(\{a_n\}) m^{-1/\alpha}(\{a_n\})$$

with $E\{\exp(iuZ_n)\} = \exp(-|u|^\alpha)$, as follows:

$$Z_a(B) = \sum_{n=1}^{N} 1_B(a_n) m^{1/\alpha}(\{a_n\}) Z_n.$$

Proposition 2.2.1: Let $Z = (Z(B): B \in \mathfrak{J})$ be an independently scattered $S\alpha S$ random measure with $0 < \alpha < 2$ and control measure $m = m_a + m_d$, and let $S = (S(B); B \in \mathfrak{J})$ be a set function. Then the following are equivalent:

i) S is an admissible translate of Z,

ii) S is an admissible translate of Z_a,

iii) S is concentrated on A, i.e.

$$S(B) = \sum_{n=1}^{N} S(\{a_n\}) 1_B(a_n),$$

and

$$\sum_{n=1}^{N} |S(\{a_n\})|^2 / m^{2/\alpha}(\{a_n\}) < \infty.$$

Furthermore a translate which is not admissible is singular.

Proof: Let ζ_a and ζ_d be the stochastic processes with parameter set $\mathfrak{J}$ defined on the probability space $(\overline{\mathbf{X}}^{\mathfrak{J}}, \mathcal{C}(\overline{\mathbf{X}}^{\mathfrak{J}}), \mu_Z)$ by

$$\zeta_a(B,x) = x(A\cap B), \text{ and } \zeta_d(B,x) = x(A^c\cap B), \quad x \in \mathbf{X}^{\mathfrak{J}}, \ B \in \mathfrak{J}.$$

Clearly

(2.2.1) $\quad \zeta_a(B, Z(\cdot,\omega)) = Z(A\cap B, \omega) = Z_a(B, \omega)$, and

$\quad\quad\quad \zeta_d(B, Z(\cdot,\omega)) = Z(A^c\cap B, \omega) = Z_d(B, \omega)$, a.s. (P),

so that ζ_a and ζ_d are independently scattered SαS random measures with control measures m_a and m_d respectively. Let ζ_a and ζ_d also denote the corresponding linear maps $x \to \zeta_a(\cdot,x)$ and $x \to \zeta_d(\cdot,x)$ from $\overline{\mathbf{X}}^{\mathfrak{J}}$ into $\overline{\mathbf{X}}^{\mathfrak{J}}$.

i) $\Rightarrow$ ii) Suppose $\mu_{S+Z} \ll \mu_Z$. Hence by Proposition 2.1.4, $S \in \mathbb{F}$ and by definition of $\mathbb{F}$ the map $F:\mathcal{L}(Z) \to \mathbf{X}$ defined by

$$F(\textstyle\sum_{k=1}^n a_k Z(A_k)) = \sum_{k=1}^n a_k S(A_k)$$

is a well defined linear functional so that S is a signed measures on $\mathfrak{J}$. Furthermore since

$$|S(B)| \leq C_{p,\alpha}\|S\|_{\mathbb{F}}\|Z(B)\|_\alpha = C_{p,\alpha}\|S\|_{\mathbb{F}}[m(B)]^{1/\alpha},$$

S is absolutely continuous with respect to m, i.e. $S(B) = \int_B \overline{z}\,dm$ for some z locally in $L_1(m)$: $z1_B \in L_1(m)$ for all $B \in \mathfrak{J}$.

It follows that $\mu_{S+Z}\zeta_d^{-1} \ll \mu_Z\zeta_d^{-1}$ or equivalently $\zeta_d(\cdot, S)$ is an admissible

translate of the process ζ_d, since ζ_d is linear. Now

$$\zeta_d(B,S) = S(A^c \cap B) = \int_{A^c \cap B} \bar{z}\,dm = \int_B \bar{z}\,dm_d \triangleq S_d(B).$$

Since m_d is nonatomic it follows from a well known results (Halmos (1975), p. 174) that we can find measurable partitions $\{B_{j,k}(B):k = 1,2,...k_j\}$, $j = 1,2,...$, of B for which

$$(2.2.2) \qquad \max_{1 \leq k \leq K_j} m_d(B_{j,k}(B)) \to 0 \qquad\qquad \text{as } j \to \infty.$$

For notational simplicity we will omit in the following the dependence on B. It follows that the triangular system of rowwise independent random variables $\{\zeta_d(B_{j,k}); k = 1,2,...,K_j, j = 1,2,...\}$ is infinitesimal, i.e. for every $\epsilon > 0$,

$$\max_{1 \leq k \leq K_j} \mu_Z(|\zeta_d(B_{j,k})| \geq \epsilon) \to 0 \qquad\qquad \text{as } j \to \infty.$$

Hence, since for every j, $\zeta_d(B) = \sum_{k=1}^{K_j} \zeta_d(B_{j,k})$, we have from the central limit theorem for triangular arrays and the fact that ζ_d has no Gaussian component that

$$\liminf_{\epsilon \to 0} \liminf_{j \to \infty} \mathrm{Var}\{\sum_{k=1}^{K_j} \zeta_d(B_{j,k}) 1_{(-\epsilon,\epsilon)}(|\zeta_d(B_{j,k})|)\} = 0$$

(see e.g. Araujo and Giné (1980), Theorem 4.7). Thus by Chebyshev's inequality

$$(2.2.3) \qquad \sum_{k=1}^{K_j} \zeta_d(B_{j,k}) 1_{(-\epsilon,\epsilon)}(|\zeta_d(B_{j,k})|)\} \to 0$$

in μ_Z-probability (in $L_p(\mu_Z)$, $p \in (0,\alpha)$) as $j \to \infty$ and $\epsilon \to 0$.

On the other hand, if $S_d(B) = \int_B \bar{z}\,dm_d$ and $m_d(B_{j,k}) \to 0$ as $j \to \infty$ then

$S_d(B_{j,k}) \to 0$ as j $\to \infty$, and hence for j large

$$S_d(B) = \sum_{k=1}^{K_j} S_d(B_{j,k}) = \sum_{k=1}^{K_j} S_d(B_{j,k}) 1_{(-\epsilon,\epsilon)}(|S_d(B_{j,k})|).$$

Similarly

$$(2.2.4) \qquad \sum_{k=1}^{K_j}(S_d(B_{j,k}) + \zeta_d(B_{j,k}))1_{(-\epsilon,\epsilon)}(|S_d(B_{j,k})) + \zeta_d(B_{j,k})|) \to S_d(B).$$

in μ_Z-probability as j $\to \infty$ and $\epsilon \to 0$.

Define for B $\in \mathfrak{J}$ the map $\phi(B,\cdot): \mathbf{X}^{\mathfrak{J}} \to \mathbf{X}^{\mathfrak{J}}$ by

$$(2.2.5) \qquad \phi(B,x) = \liminf_{\epsilon \to 0} \liminf_{j \to \infty} \sum_{k=1}^{K_j} x(B_{j,k}) 1_{(-\epsilon,\epsilon)}(|x(B_{j,k})|).$$

Suppose S_d is not identically zero. Then there exists B $\in \mathfrak{J}$ such that $S_d(B) \neq 0$. It follows from (2.2.3) and (2.2.4) that

$$\phi(B,\zeta_d(\cdot,x)) = 0 \text{ and } \phi(B,S_d+\zeta_d(\cdot,x)) = S_d(B) \text{ a.e. } (\mu_Z)$$

Thus $\mu_{S+Z}\phi^{-1}(B,\cdot) \perp \mu_Z\phi^{-1}(B,\cdot)$ and hence $\mu_{S+Z} \perp \mu_Z$ which is a contradiction. Therefore $S_d(B) = \int_B \bar{z}dm_d = 0$ for all B $\in \mathfrak{J}$, i.e. z = 0 a.e. (m_d), so that

$$(2.2.6) \qquad S(B) = \int_B \bar{z}dm_a = \sum_{n=1}^{N} \bar{z}(a_n)1_B(a_n)m(\{a_n\}).$$

Reasoning as before we have $\mu_{S+Z}\zeta_a^{-1} \ll \mu_Z\zeta_a^{-1}$, i.e. $\zeta_a(\cdot, S)$ is an admissible translate of ζ_a (or Z_a), and by (2.2.6)

$$\zeta_a(B,S) = S(A \cap B) = \int_{A \cap B} \bar{z}dm = \int_B \bar{z}dm_a = S(B)$$

i.e. $S = \zeta_a(\cdot, S)$ is an admissible translate of Z_a.

ii) $\Rightarrow$ i). Suppose S is an admissible translate of Z_a. Since $Z = Z_a + Z_d$ and Z_a and Z_d are independent we have $\mu_Z = \mu_{Z_a} * \mu_{Z_d}$. Then $\mu_{S+Z_a} \ll \mu_{Z_a}$ implies $\mu_{S+Z} \ll \mu_Z$. Indeed

$$0 = \mu_Z(B) = \int_{\overline{X}^{\mathfrak{I}}} \mu_{Z_a}(B-x)\mu_{Z_d}(dx)$$

implies

$$\mu_{Z_a}(B-x) = 0 \quad \text{a.e. } (\mu_{Z_d})$$

hence

$$0 = \mu_{S+Z_a}(B-x) = \mu_{Z_a}(B-S-x) \quad \text{a.e. } (\mu_{Z_d})$$

and thus

$$\mu_{S+Z}(B) = \mu_Z(B-S) = \int_{\overline{X}^{\mathfrak{I}}} \mu_{Z_a}(B-S-x)\mu_{Z_d}(dx) = 0.$$

ii) $\Rightarrow$ iii). Because $S \in \mathbb{F}$, S is absolutely continuous with respect to m_d, $S(B) = \sum_{n=1}^{N} S(\{a_n\})1_B(a_n)$. Let $\psi: \overline{X}^{\mathfrak{I}} \to \overline{X}^{\mathcal{N}}$, where $\mathcal{N} = \{1,...,N\}$ if $N < \infty$ and $\mathcal{N} = \mathbb{N}$ otherwise, be defined by

$$[\psi(x)](n) = \psi(n,x) = \zeta_a(\{a_n\}, x)/m^{1/\alpha}(\{a_n\}).$$

Thus by (2.2.1), $\psi(n,\cdot)$, $n \in \mathcal{N}$, are standard $S\alpha S$ i.i.d. random variables,

$$\psi(n,S) = \zeta_a(\{a_n\}, S)/m^{1/\alpha}(\{a_n\}) = S(\{a_n\})/m^{1/\alpha}(\{a_n\})$$

and

$$\psi(n,S+x) = \psi(n,S)+\psi(n,x) = S(\{a_n\})/m^{1/\alpha}(\{a_n\})+\psi(n,x).$$

Now $\mu_{S+Z_a} \ll \mu_{Z_a}$ implies $\mu_{S+Z_a}\psi^{-1} \ll \mu_{Z_a}\psi^{-1}$, i.e.

$(S(a_n)/m^{1/\alpha}(\{a_n\}); n \in \mathcal{N})$ is an admissible translate of the random element $(\psi(n,\cdot); n \in \mathcal{N})$ defined on the probability space $(\overline{\mathbf{X}}^{\mathfrak{J}}, \mathcal{C}(\overline{\mathbf{X}}^{\mathfrak{J}}), \mu_Z)$. It follows from Shepp (1965) if $N = \infty$ and trivially if $N < \infty$ that $\sum_{n=1}^{N} S^2(\{a_n\})/m^{2/\alpha}(\{a_n\}) < \infty$.

iii) $\Rightarrow$ ii). Conversely, if $\sum_{n=1}^{N} S^2(\{a_n\})/m^{2/\alpha}(\{a_n\}) < \infty$ it follows from Shepp (1965) and the fact that stable densities have finite Fisher information (DuMouchel (1973)) that $(S(\{a_n\})/m^{1/\alpha}(\{a_n\}); n \in \mathcal{N})$ is an admissible translate of $(\psi(n,\cdot); n \in \mathcal{N})$ (the result is trivial if $N < \infty$). Therefore

$$\sum_{n=1}^{N} S(\{a_n\})1_B(\{a_n\}) = S(B)$$

is an admissible translate of the process

$$\sum_{n=1}^{N} 1_B(\{a_n\})m^{1/\alpha}(\{a_n\})\psi(n,x) = \sum_{n=1}^{N} 1_B(a_n)\zeta(\{a_n\},x) = \zeta_a(B,x)$$

and hence of Z_a.

To prove that a translate S which is not admissible is singular it suffices to consider such a translate in $\mathbf{F}$, i.e. from the proof of i) $\Rightarrow$ ii), $S(B) = \int_B \bar{z}\,dm$. If $m_d(|z| > 0) > 0$ then $\mu_{S+Z} \perp \mu_Z$. Thus assume

$$S(B) = \int_B \bar{z}\,dm_a = \sum_{n=1}^{N} S(\{a_n\})1_B(a_n).$$

Since it is not admissible, by iii) $N = \infty$ and $\sum_{n=1}^{N} S^2(\{a_n\})/m^{2/\alpha}(\{a_n\}) = \infty$. Hence from Shepp (1965), $(S(\{a_n\})/m^{1/\alpha}(\{a_n\}); n \in \mathbf{N})$ is a singular translate of $(\psi(n,\cdot); n \in \mathbf{N})$, i.e. $\mu_{S+Z}\psi^{-1} \perp \mu_Z\psi^{-1}$, which implies $\mu_{S+Z} \perp \mu_Z$. $\qquad\square$

It follows that the admissible translates of a SαS independently scattered random measure are quite different in the Gaussian and non-Gaussian cases. Indeed, for Z

Gaussian ($\alpha = 2$) every element in its function space (i.e. its RKHS)

$$\mathbf{F}_2 = \{S; S(B) = \int_B \bar{z} dm, \ \bar{z} \in L_2(m)\}$$
$$= \{S; S \text{ signed measure on } \sigma(\mathfrak{I}), \ S \ll m, \frac{dS}{dm} \in L_2(m)\}$$

(see, e.g. Chatterji and Mandrekar (1978)) is an admissible translate, while, e.g. for Z non-Gauss'an with $1 < \alpha < 2$ its only admissible translates are $S(B) = \int_B \bar{z} dm$, $\bar{z} \in L_{\alpha*}(m)$, with $\bar{z} = 0$ a.e. (m_d), and $\sum_{n=1}^N |S(\{a_n\})|^2 / m^{2/\alpha}(\{a_n\}) < \infty$. Hence for $1 < \alpha < 2$ the set of admissible translates is a proper subset of the function space $\mathbf{F}_\alpha$ which is given by

$$\mathbf{F}_\alpha = \{S: s \text{ signed measure on } \sigma(\mathfrak{I}), \ S \ll m, \frac{dS}{dm} \in L_{\alpha*}(m)\}.$$

In particular, while a diffuse Gaussian random measure has a rich class of admissible translates, a diffuse non-Gaussian SαS random measure has no admissible translate whatever. On the other hand, if m (or Z) is atomic ($m_d = 0$), the condition in Proposition 2.2.1 iii) extends the Gaussian condition. Indeed if $\alpha = 2$ and

$$S(B) = \int_B \frac{dS}{dm} dm = \sum_{n=1}^N \frac{dS}{dm}(a_n) m(\{a_n\})$$

then $\sum_{n=1}^N |S(\{a_n\})|^2 / m(\{a_n\})^{2/\alpha} < \infty$ is equivalent to $\frac{dS}{dm} \in L_2(m)$.

The results of Proposition 2.2.1 can now be used to obtain a dichotomy for the translates of an invertible SαS process, and to characterize its admissible translates as those of its atomic component. In order to state the result for a SαS process X with spectral representation $X(t) = \int_I f(t,u) Z(du)$ and control measure m, we introduce the independent SαS diffuse and atomic component processes of X:

$$X_d(t) = \int_A cf(t,u)Z(du) = \int_I f(t,u)Z_d(t),$$

$$X_a(t) = X(t) - X_d(t) = \int_A f(t,u)Z(du) = \int_I f(t,u)Z_a(du).$$

The atomic component X_a has a series expansion

$$X_a(t) = \sum_{n=1}^N f(t,a_n)Z(\{a_n\}),$$

which can be normalized by putting

$$Z_n = Z(\{a_n\})/m^{1/\alpha}(\{a_n\}) \quad \text{and} \quad f_n(t) = f(t,a_n)m^{1/\alpha}(\{a_n\}),$$

so that the Z_n's are standard SαS i.i.d. random variables, for all $t \in T$, $\sum_{n=1}^N |f_n(t)|^\alpha < \infty$, and

$$X_a(t) = \sum_{n=1}^N f_n(t)Z_n.$$

Proposition 2.2.2. Let $X = (X(t); t \in T)$ be a SαS process with $0 < \alpha < 2$, invertible spectral representation $X(t) = \int_I f(t,u)Z(du)$ and control measure m, and let $s = (s(t); t \in T)$ be a function on T. Then the following are equivalent:

   i) s is an admissible translate of X,

   ii) s is an admissible translate of X_a,

   iii) $s(t) = \sum_{n=1}^N s'_n f(t,a_n)$ with $\sum_{n=1}^N |s'_n|^2/m^{2/\alpha}(\{a_n\}) < \infty$,

i.e.

$$s(t) = \sum_{n=1}^N s_n f_n(t) \quad \text{with} \quad \sum_{n=1}^N |s_n|^2 < \infty.$$

Furthermore a translate which is not admissible is singular.

Proof: i) Since $1_B \in L_\alpha(m) = \mathcal{L}(f)$, for any $B \in \mathfrak{I}$, there exist

$\phi_n(B,\cdot) \in \text{sp}\{(f(t,\cdot); t \in \mathbf{T})\}$, $n = 1,2,...$, i.e.

$$\phi_n(B,\cdot) = \sum_{n=1}^{N_n(B)} a_{n,k}(B) f(t_{n,k}(B),\cdot),$$

such that

$$\phi_n(B,\cdot) \to 1_B(\cdot) \text{ in } L_\alpha(m) \qquad\qquad \text{as } n \to \infty.$$

Define

$$\phi_n(B,x) = \sum_{k=1}^{N_n(B)} a_{n,k}(B) \, x(t_{n,k}(B)), \quad x \in \mathbf{X}^\mathbf{T}.$$

Thus

$$(2.2.7) \qquad \phi_n(B,X(\cdot,\omega)) = \sum_{n=1}^{N_n(B)} a_{n,k}(B) X(t_{n,k}(B),\omega)$$

$$= \int_{\mathbb{I}} \phi_n(B,u) Z(du,\omega) \to \int_{\mathbb{I}} 1_B(u) Z(du,\omega) = Z(B,\omega)$$

in L_p (hence in probability) as $n \to \infty$. Thus $(\phi_n(B,\cdot); n \in \mathbf{N})$ converges in μ_X-measure. Let $(\phi_{n_k}(B,\cdot); k \in \mathbf{N})$ be a subsequence converging a.e. (μ_X) and define

$$\check{Z}(B) = \check{Z}(B,\cdot) = \liminf_{k \to \infty} \phi_{n_k}(B,\cdot) \, 1_{\{x;\phi_{n_k}(B,x) \text{ converges}\}}(\cdot).$$

$\check{Z}(B,\cdot)$ is a well defined $\mathcal{C}$-measurable function on $\mathbf{X}^\mathbf{T}$ for each $B \in \mathfrak{I}$. Hence $\check{Z} = (\check{Z}(B): B \in \mathfrak{I})$ is a stochastic process on the probability space $(\mathbf{X}^\mathbf{T}, \mathcal{C}, \mu_X)$, and from (2.2.7), $\check{Z}(B,X(\cdot,\omega)) = Z(B,\omega)$ a.s., so that $\check{Z}$ is equal in law to Z. i.e. $\check{Z}$ is an independently scattered SαS random measure with control measure m.

i) $\Rightarrow$ ii) Let s be an admissible translate of X. From Proposition 2.1.4, $s \in \mathbb{F}$, i.e.

for $p \in (0,\alpha)$,

$$|\sum_{k=1}^{n} a_k s(t_k)| \leq \|s\|_{\mathbf{F}} \|\sum_{k=1}^{n} a_k X(t_k)\|_{L_p(P)}.$$

Hence as in Proposition 2.2.1, $F[\sum_{k=1}^{n} a_k X(t_k)] = \sum_{k=1}^{n} a_k s(t_k)$ is a well defined continuous linear functional on $\mathcal{L}(X)$ and $s(t) = F[X(t)]$. Thus

$$\phi_n(B,s) = \sum_{k=1}^{N_n(B)} a_{n,k}(B) s(t_{n,k}(B))$$

$$= F[\sum_{k=1}^{N_n(B)} a_{n,k}(B) X(t_{n,k}(B)] \rightarrow F[Z(B)] \qquad \text{as } n \rightarrow \infty.$$

Hence for all $B \in \mathfrak{I}$,

(2.2.9) $\quad \tilde{Z}(B,s) = F(Z(B))$

and

(2.2.10) $\quad \tilde{Z}(B,s + x) = \tilde{Z}(B,s) + \tilde{Z}(B,x)$.

Now if $\tilde{Z}_d(B,\cdot) = \tilde{Z}(A^c \cap B,\cdot)$, then $\tilde{Z}_d = (\tilde{Z}_d(B,\cdot); B \in \mathfrak{I})$ is an independently scattered $S\alpha S$ random measure with control measure m_d and by (2.2.10) it has $\tilde{Z}_c(\cdot,s)$ as an admissible translate. But m_d is non-atomic, thus by Proposition 2.2.1, $\tilde{Z}_d(\cdot,s) = 0$, i.e. for all $B \in \mathfrak{I}$,

$$0 = \tilde{Z}_d(B,s) = \tilde{Z}(A^c \cap B,s) = F(Z(A^c \cap B)) = F(Z_d(B)),$$

and hence

$$s(t) = F[X(t)] = F[X_a(t) + X_d(t)] = F[X_a(t)]$$

(since X_d is obtained by a linear operation on Z_d which implies $F[X_d(t)] = 0$).

Therefore

$$(2.2.11) \quad s(t) \; = \; F[X_a(t)] = F[\textstyle\sum_{n=1}^{N} f_n(t) \, Z_n]$$

$$= \; \textstyle\sum_{n=1}^{N} f_n(t) F(Z_n) \; = \; \textstyle\sum_{n=1}^{N} f_n(t) s_n$$

$$= \; \textstyle\sum_{n=1}^{N} f_n(t,a_n) \, s_n',$$

where $s_n \; = \; F(Z_n)$ and $s_n' = m^{1/\alpha}(\{a_n\}) s_n$. On the other hand

$$\tilde{X}_a \; = \; (\tilde{X}_a(t,x) \; = \; \textstyle\sum_{n=1}^{N} f(t,a_n) \tilde{Z}(\{a_n\},x); \; t \in T)$$

has distribution μ_{X_a} and by the linearity of the map $x \to \tilde{X}_a(\cdot,x)$, the function $\tilde{X}_a(\cdot,s)$ is an admissible translate of $\tilde{X}$ and hence of X_a. But

$$\tilde{X}_a(t,s) \; = \; \textstyle\sum_{n=1}^{N} f(t,a_n) \tilde{Z}(\{a_n\},s)$$

$$= \; \textstyle\sum_{n=1}^{N} f(t,a_n) F[Z(\{a_n\})]$$

$$= \; \textstyle\sum_{n=1}^{N} f(t,a_n) s_n' \; = \; \textstyle\sum_{n=1}^{N} f_n(t) s_n \; = \; s(t),$$

i.e. s is an admissible translate of X_a.

$\underline{ii) \Rightarrow i)}$ The proof is identical to that in Proposition 2.2.1.

$\underline{ii) \Rightarrow iii)}$ The proof is as in Proposition 2.2.1, with $v_n(n,x) = \tilde{Z}(\{a_n\})/m^{1/\alpha}(\{a_n\})$, so that by (2.2.9)

$$(2.2.12) \quad v(n,s) \; = \; \tilde{Z}(\{a_n\}, s)/m^{1/\alpha}(\{a_n\})$$

$$= \; F[Z(\{a_n\})]/m^{1/\alpha}(\{a_n\})$$

$$= \; s_n'/m^{1/\alpha}(\{a_n\}) \; = \; s_n.$$

To prove that a translate which is not admissible is singular, it suffices to consider

$s \in \mathsf{F}(X)$, i.e., $s(t) = \mathsf{F}[X(t)]$, as by Proposition 2.1.4, $s \notin \mathsf{F}$ implies singularity.

Suppose $\mathsf{F}[X_d(t)] \neq 0$. Then there exists $B \in \mathfrak{I}$ such that $\mathsf{F}[Z_d(B)] \neq 0$ and by (2.2.9),

$$\tilde{Z}_d(B,s) = \tilde{Z}(A^c \cap B,s) = \mathsf{F}[Z(A^c \cap B)] = \mathsf{F}[Z_d(B)] \neq 0.$$

It follows from Proposition 2.2.1 that $\mu_{s+X}\tilde{Z}_d^{-1} \perp \mu_X\tilde{Z}_d^{-1}$ and hence $\mu_{s+X} \perp \mu_X$. Therefore $s(t) = \mathsf{F}[X_a(t)] = \sum_{n=1}^{N} f_n(t)s_n$ and as in the proof of proposition 2.2.1, $\sum_{n=1}^{N} |s_n|^2 = \infty$ implies $\mu_{s+X} \perp \mu_X$. □

It follows from Proposition 2.2.2 that for an invertible SαS process with nonatomic control measure every non-zero translate is singular. In particular, this contains Corollary 10.1 of Zinn (1975). Applied to SαS processes with purely atomic control measure, Proposition 2.2.2 is a stochastic process version of a result proved in Thang and Tien (1979), Theorem 4, for SαS measures with discrete spectral measures on separable Banach spaces. The proposition completes the result in Thang and Tien (1979) providing a dichotomy for the problem of admissible translates.

Proposition 2.2.2 also provides examples where the set of admissible translates is a non-trival proper subset of the function space F of the process X. E.g. if $X(t) = \sum_{n=1}^{\infty} f_n(t)Z_n$, $t \in \mathsf{T}$, where $Z_1, Z_2,...$ are i.i.d. standard SαS random variables with $1 < \alpha \leq 2$ and $L_\alpha\text{-}\overline{sp}\{(f_n(t); n \in \mathsf{N}); t \in \mathsf{T}\} = l_\alpha$, then

$$\mathsf{F}_\alpha = \{s: s(t) = \sum_{n=1}^{\infty} s_n f_n(t), \sum_{n=1}^{\infty} |s_n|^{\alpha^*} < \infty\},$$

while the set of admissible translates is the infinite dimensional subspace (since $\alpha^* \geq 2$) of F_α for which $\sum_{n=1}^{\infty} |s_n|^2 < \infty$; hence we have equality only if $\alpha = 2$, and proper inclusion if $1 < \alpha < 2$. There is a natural identification between the set of admissible translates, which is always a linear space, and the Hilbert space l_2, namely

$$(s_n; n \in \mathbf{N}) \quad \rightarrow \quad s(\cdot) = \sum_{n=1}^{\infty} s_n f_n(\cdot).$$

This map is invertible with the inverse map given by the transformation ψ defined in the proof in Proposition 2.2.2 restricted to the set of admissible translates of X (cf. 2.2.12). Thus for every $\alpha \in (0,2)$, the linear space of admissible translates can be given a Hilbert space structure by defining the inner product

$$<s_1,s_2> \quad = \quad <(s_{1,n}),(s_{2,n})>_{l_2} \quad = \quad \sum_{n=1}^{\infty} s_{1,n} s_{2,n},$$

where $s_i(t) = \sum_{n=1}^{\infty} s_{i,n} \, f_n(t)$, $i = 1,2$. Note that in this case when $1 < \alpha < 2$, $\|s\|_{\mathbf{F}} = (\sum |s_n|^{\alpha^*})^{1/\alpha^*}$ and hence $\|\cdot\|_{\mathbf{F}}$ is not a natural norm on the linear space of admissible translates, in contrast with the case of Gaussian ($\alpha = 2$) and α-sub-Gaussian processes with $1 < \alpha < 2$.

Important examples of SαS processes with invertible spectral representation are presented in the following.

Harmonizable SαS processes (and sequences).

Let $X = (X(t); t \in \mathbf{T})$, $\mathbf{T} = \mathbf{R}^d$ or $\mathbf{Z}^d$, $d \in \mathbf{N}$, be a SαS harmonizable process, i.e., X has the representation

$$X(t) \quad = \quad \int_{\mathbf{I}} e^{i<t,u>} Z(du), \quad t \in \mathbf{T},$$

where $\mathbf{I} = \mathbf{R}^d$ and $[-\pi,\pi]^d$ for $\mathbf{T} = \mathbf{R}^d$ and $\mathbf{Z}^d$ respectively and Z is a SαS independently scattered random measure with finite control measure m, referred to as the spectral measure of the harmonizable process X. If the spectral measure m is nonatomic and $0 < \alpha < 2$ then it follows from Proposition 2.2.2 that X has no

nontrivial admissible translate. When the stable distribution of Z is radially symmetric, i.e. when X is stationary, this result exhibits a different behavior compared with the stationary Gaussian processes $\alpha = 2$, whose admissible translates are precisely the functions

$$s(t) = \int_\mathbb{I} e^{i<t,u>} \bar{z}(u)m(du), \ z \in L_2(m).$$

In contrast, if m is purely discrete, i.e. X has a Fourier series representation

$$X(t) = \sum_{n=1}^N b_n e^{i<c_n,t>} Z_n, \quad N \leq \infty,$$

with Z_n's i.i.d. standard SαS random variables and $\sum_{n=1}^\infty |b_n|^\alpha < \infty$, the set of admissible translates is

$$\{s: s(t) = \sum_{n=1}^N s_n e^{i<c_n,t>} ; \ \sum_{n=1}^N |s_n/b_n|^2 < \infty\},$$

and depends on α, $0 < \alpha \leq 2$, only via the sequence $(b_n; n \in \mathbb{N}) \in l_\alpha$. In other words for fixed $(b_n; n \in \mathbb{N}) \in l_\beta$, $1 \leq \beta \leq 2$, define

$$X_\alpha(t) = \sum_{n=1}^\infty b_n \exp\{i<c_n,t>\}Z_{n,\alpha} ,$$

where the $Z_{n,\alpha}$'s are standard i.i.d. SαS with $\beta \leq \alpha \leq 2$ for $1 < \beta \leq 2$ and $1 < \alpha \leq 2$ for $\beta = 1$. Then all these processes X_α have the same set of admissible translates.

Continuous time SαS moving averages.

Another class of SαS processes is the class of real moving averages,

$$X(t) = \int_{\mathbf{R}} f(t-u)Z(du), \quad t \in \mathbf{R},$$

where Z has Lebesgue control measure and $f \in L_\alpha(\text{Leb})$. When f vanishes on the negative line, they are called nonanticipating moving averages and they occur as the stationary solutions of nth order linear stochastic differential equations with constant coefficients driven by stable motion Z.

In the Gaussian case $\alpha = 2$ the admissible translates coincide with the function space (RKHS)

$$\begin{aligned}
\mathbf{F}_2 &= \{s: s(t) = \int_{\mathbf{R}} f(t-u)z(u)du, \, z \in L_2(\text{Leb})\} \\
&= \{s: s \in L_2(\text{Leb}), \, s/\hat{f} \in L_2(\text{Leb})\},
\end{aligned}$$

where $\hat{\ }$ denotes Fourier transform.

Examples of moving averages with invertible spectral representation and therefore with no admissible translates, can be obtained by taking

i) f continuous and equal to zero on $(-\infty,0)$ and at infinity (Atzman (1983), Theorem 2),

ii) $\alpha \in (1,2)$ and f the Fourier transform of some function F in $L_{\alpha*}(\text{Leb})$ with $F \neq 0$ a.e. (Leb) (Titchmarsh (1928), Theorem 75).

Case i) includes nonanticipating moving averages with continuous kernel f, while case ii) contains certain nonanticipating moving averages with discontinuous kernels f, namely the stationary solutions of nth order linear stochastic differential equations with constant coefficients. There $f(t)$ is a linear combination of functions of the type $t^{k-1}e^{-at}1_{(0,\infty)}(t)$ with $k \in \mathbf{N}$ and $a > 0$, which are Fourier transforms of the $L_{\alpha*}(\text{Leb})$ functions $\Gamma(k)/[2\pi(a+iu)]$. Hence f is the Fourier transform of an $L_{\alpha*}(\text{Leb})$ function which is not zero a.e. (Leb) so that $\overline{\text{sp}}\{f(t-\cdot); t \in \mathbf{R}\} = L_\alpha(\text{Leb})$, i.e. X is invertible. Thus solutions of nth order stochastic differential equations driven by $S\alpha S$

motion have no admissible translate for $1 < \alpha < 2$. This is in sharp contrast with the Gaussian case $\alpha = 2$. E.g., if $n = 1$, $f(t) = e^{-t}1_{(0,\infty)}(t)$ and the stable Ornstein-Uhlenbeck (OU) process

$$X(t) \;=\; \int_{-\infty}^{t} e^{-(t-u)}Z(du), \; t \in \mathbf{R},$$

has no admissible translates for $1 < \alpha < 2$, while for $\alpha = 2$ all translates of the form

$$s(t) \;=\; \int_{-\infty}^{t} e^{-(t-u)}z(u)du, \quad z \in L_2(\text{Leb}) \text{ and } t \in \mathbf{R},$$

are admissible for the OU process X.

Discrete time SαS processes (SαS sequences) with invertible spectral representation have similar sets of admissible translates in the Gaussian and non-Gaussian stable case. Of course nonadmissible translates are singular.

Independent sequences and partial sums of independent SαS random variables

The set of admissible translates of a sequence of independent SαS random variables $X = (X_n; n \in \mathbf{N})$ is given by

$$\{s \;=\; (s_n; n \in \mathbf{N}); \; \sum_{n=1}^{\infty} (s_n/\|X_n\|_\alpha)^2 \; < \; \infty\}.$$

The set of admissible translates of a sequence $(Y_n = \sum_{k=1}^{n} X_k; n \in \mathbf{N})$ of partial sums of independent SαS random variables X_k is

$$\{s \;=\; (s_n; n \in \mathbf{N}); \; \sum_{n=1}^{\infty} (s_n-s_{n-1})^2/\|X_n\|_\alpha^2 \; < \; \infty, s_0 \;=\; 0\}.$$

Mixed auto-regressive moving averages of order (p,q) (ARMA (p,q)).

Let $X = (X_n; n \in \mathbf{N})$ be defined by the difference equation

$$X_n - a_1 X_{n-1} - \ldots - a_p X_{n-p} = Z_n + b_1 Z_{n-1} + \ldots + b_q Z_{n-q}$$

where $Z = (Z_n; n \in \mathbf{N})$ is a sequence of i.i.d. standard SαS random variables. If the polynomials $P(u) = 1 - a_1 u - \ldots - a_p u^p$ and $Q(u) = 1 + b_2 u + \ldots b_q u^q$ satisfy the condition $P(u)Q(u) \neq 0$ for all $u \in \mathbf{C}$ with $|u| \leq 1$, then the difference equation defining X has a unique stationary solution of the moving average form

$$X_n = \sum_{k=-\infty}^{n} g_{n-k} Z_k,$$

and in addition

$$Z_n = X_n - \sum_{j=1}^{\infty} h_j X_{n-j}$$

(see e.g. Cline and Brockwell (1985)). The coefficients $\{g_n; n \in \mathbf{N}\}$ and $\{h_n; n \in \mathbf{N}\}$ are uniquely determined by the power series expansions

$$Q(u)/P(u) = \sum_{j=0}^{\infty} g_j u^j \quad \text{and} \quad P(u)/Q(u) = 1 - \sum_{j=1}^{\infty} h_j u^j, \ |u| \leq 1.$$

Thus $\mathcal{L}(X) = \mathcal{L}(Z)$, i.e. X is invertible, and hence, by Proposition 2.2.2., $s = (s_n; n \in \mathbf{Z})$ is an admissible translate of X if and only if it is of the form

$$s_n = \sum_{k=-\infty}^{n} g_{n-k} z_k$$

where $\sum_{k=-\infty}^{\infty} z_k^2 < \infty$.

We should note the different behavior of moving averages in continuous and in discrete time. A continuous time moving average may have no admissible translates, whereas a discrete time ARMA sequence has a set of admissible translates identical to

the Gaussian case. The difference will be in the form of the Radon-Nikodym derivatives.

2.3 Comments on the Radon-Nikodym derivatives

Expressions for Radon-Nikodym derivatives in the non-Gaussian stable case are difficult to obtain even in the case of invertible processes since no analytic expression is generally available for the SαS densities.

As observed in Section 2.2 the measures μ_X and μ_{s+X} are convolutions of the measures μ_{X_a} with μ_{X_d} and μ_{s+X_a} with μ_{X_d} respectively, or in other words if $\Sigma : X^T \times X^T \to X^T$ is the map $\Sigma(x_1, x_2) = x_1 + x_2$ then $\mu_X = (\mu_{X_a} \times \mu_{X_d})\Sigma^{-1}$ and $\mu_{s+X} = (\mu_{s+X_a} \times \mu_{X_d})\Sigma^{-1}$. When $\mu_{s+X} \ll \mu_X$, we have the following lemma is useful

__Lemma 2.3.1__ Let $(\Omega_i, \mathcal{F}_i)$, $i = 1,2$, be measurable spaces and let the map $T : \Omega_1 \to \Omega_2$ be measurable. If μ_1 and μ_2 are probability measures on $(\Omega_1, \mathcal{F}_1)$ such that $\mu_2 \ll \mu_1$, then

$$\frac{d\mu_2 T^{-1}}{d\mu_1 T^{-1}} \circ T = E_{\mu_1}\left(\frac{d\mu_2}{d\mu_1}\bigg/ T\right) \quad \text{a.e. } (\mu_1)$$

i.e.

$$\frac{d\mu_2 T^{-1}}{d\mu_1 T^{-1}}(\omega) = E_{\mu_1}\left(\frac{d\mu_2}{d\mu_1}\bigg/ T = \omega\right) \quad \text{a.e. } (\mu_1 T^{-1})$$

__Proof.__ Clearly $\mu_2 T^{-1} \ll \mu_1 T^{-1}$ and for all $B \in \mathcal{F}_2$ we have

$$\int_{T^{-1}(B)} \frac{d\mu_2 T^{-1}}{d\mu_1 T^{-1}} \circ T \, d\mu_1 = \int_B \frac{d\mu_2 T^{-1}}{d\mu_1 T^{-1}} \, d\mu_1 T^{-1}$$

$$= \mu_2(T^{-1}B) = \int_{T^{-1}(B)} \frac{d\mu_2}{d\mu_1} \, d\mu_1. \qquad \square$$

It follows from Lemma 2.3.1 that if $\mu_{s+X} \ll \mu_X$ then a.e. (μ_X)

$$
\begin{aligned}
\frac{d\mu_{s+X}}{d\mu_X}(x) &= E_{\mu_X}\left(\frac{d(\mu_{s+X_a} \times \mu_{X_d})}{d(\mu_{X_a} \times \mu_{X_d})} \middle/ \Sigma = x\right) \\
&= E_{\mu_X}\left(\frac{d\mu_{s+X_a}}{d\mu_{X_a}} \middle/ \Sigma = x\right)
\end{aligned}
$$

since

$$
\frac{d\mu_{s+X_a} \times \mu_{X_d}}{d\mu_{X_a} \times \mu_{X_d}}(x_1, x_2) = \frac{d\mu_{s+X_a}}{d\mu_{X_a}}(x_1)\frac{d\mu_{X_d}}{d\mu_{X_d}}(x_2) = \frac{d\mu_{s+X_a}}{d\mu_{X_a}}(x_1).
$$

The determination of the above conditional expectation is not possible in general. However if the process is invertible and has only atomic component, the invertibility of the process allows the representation of the Radon-Nikodym derivation in terms of a standard SαS density as shown in the following

Proposition 2.3.2. Let $X = (X(t); t \in T)$ be a SαS process with $\alpha \in (0,2)$, invertible spectral representation and discrete control measure. If s is an admissible translate of X, then a.e. (μ_X)

$$
\frac{d\mu_{s+X}}{d\mu_X}(x) = \prod_{n=1}^{N}\left(\frac{f(\Psi(n,x) - \Psi(n,s))}{f(\Psi(n,x))}\right), \quad x \in X^T,
$$

where Ψ is defined in (2.2.12) and f is the standard SαS density.

Proof. Let $\Psi: X^T \to X^{\{1,2,\ldots,n\} \cap N}$ be the map defined in (2.2.12), i.e.

$$
\Psi(x) = (\Psi(n,x) = \tilde{Z}(\{a_n\},x)/m^{1/2}(\{a_n\}); m \in \{1,2,\ldots,n\} \cap N).
$$

By lemma 2.3.1.,

$$\frac{d\mu_{s+X}\Psi^{-1}}{d\mu_X\Psi^{-1}} \circ \Psi = E_{\mu_X}\left(\frac{d\mu_{s+X}}{d\mu_X} \Big/ \Psi\right).$$

Since

$$\mu_X(\{x;\ x(t) = \sum_{n=1}^{N} f_n(t)\ \Psi(n,x)\})$$

$$= P(\{\omega;\ X(t,\omega) = \sum_{n=1}^{N} f_n(t)\ \Psi(n,X(\cdot,\omega))\})$$

$$= P(\{\omega;\ X(t,\omega) = \sum_{n=1}^{N} f_n(t)\ Z_n(\omega)\}) = 1$$

we have $\overline{C} = \overline{\sigma(\Psi)}$ (where $^{-}$ denotes the completion with respect to μ_X (μ_{s+X})), so that

$$E\left(\frac{d\mu_{s+X}}{d\mu_X} \Big/ \Psi\right) = \frac{d\mu_{s+X}}{d\mu_X}.$$

On the other hand by Kakutani's Theorem (or trivialy if $N < \infty$),

$$\frac{d\mu_{s+X}\psi^{-1}}{d\mu_X\psi^{-1}}\ (y) = \prod_{n=1}^{N}\frac{f(y_n-\Psi(n,s))}{f(y_n)},$$

$y = (y_n;\ n \in \{1,...,n\}\cap N)$. Therefore

$$\frac{d\mu_{s+X}}{d\mu_X}\ (x) = \prod_{n=1}^{N} \frac{f(\Psi(n,x) - \Psi(n,s))}{f(\Psi(n,x))}, \quad x \in X^T. \qquad \square$$

For $x = X(\cdot,\omega)$ the Radon-Nikodym derivative can be expressed as

$$\frac{d\mu_{s+X}}{d\mu_X}(X(\cdot,\omega)) \;=\; \prod_{n=1}^{N} f\left(\frac{Z(\{a_n\},\omega)-s_n}{m^{1/\alpha}(\{a_n\})}\right) \Bigg/ f\left(\frac{Z(\{a_n\},\omega)}{m^{1/\alpha}(\{a_n\})}\right)$$

a.e. (P). where $s_n = m^{1/\alpha}(\{a_n\})\Psi(n,s)$.

When $\alpha = 2$, $f(x+s)/f(x) = \exp(sx - s^2/2)$ and thus we have the well known expression

$$\frac{d\mu_{s+X}}{d\mu_X}(x) \;=\; \exp\left[\sum_{n=1}^{\infty}(s_n x_n - s_n^2/2)\right]$$

where for $\{\phi_n; n \in \mathbf{N}\}$ a complete orthonormal system of eigenvectors of the covariance operator of X and $\{\lambda_n; n \in \mathbf{N}\}$ the corresponding eigenvalues

$$x_n \;=\; \Psi(n,x) \;=\; \lambda_n^{-1/2} \int_I x(t)\,\phi_n(t)dt,$$

$$s_n \;=\; \Psi(n,s) \;=\; \lambda_n^{-1/2} \int_I s(t)\,\phi_n(t)dt,$$

or equivalently

$$x(t) \;=\; \sum_{n=1}^{\infty} \lambda_n^{-1/2}\,\phi_n(t)x_n,$$

$$s(t) \;=\; \sum_{n=1}^{\infty} \lambda_n^{-1/2}\,\phi_n(t)s_n, \quad \Sigma|s_n|^2 \;<\; \infty .$$

For non-Gaussian stable processes explicit expressions for Ψ are not available in general. One example where this is possible is when X is a Fourier series, i.e.

$$X(t) = \sum_{n=1}^{\infty} b_n \, e^{ic_n t} Z_n, \quad \Sigma |b_n|^{\alpha} < \infty, \; t \in \mathbf{T} \, .$$

Using arguments identical to the invension theorem for Fourier transform, one can show that

$$x_n \triangleq \Psi(n,x) = b_n^{-1} \lim_{T \to \infty} \frac{1}{2\pi} \int_{-T}^{T} e^{-ic_n t} x(t) dt,$$

where the limit is in μ_X (μ_{s+X}) measure and for s an admissible translate, i.e.

$$s(t) = \sum_{n=1}^{\infty} b_n \, e^{ic_n t} s_n \, , \quad \sum_{n=1}^{\infty} |s_n|^2 < \infty \, ,$$

we have

$$s_n = b_n^{-1} \lim_{T \to \infty} \frac{1}{2\pi} \int_{-T}^{T} e^{ic_n t} s(t) dt.$$

Thus

$$\frac{d\mu_{s+X}}{d\mu_X}(x) = \prod_{n=1}^{\infty} \frac{f(x_n - s_n)}{f(x_n)} \quad \text{a.e. } (\mu_X) \, ,$$

i.e.

$$\frac{d\mu_{s+X}}{d\mu_X}(X(\cdot,\omega)) = \prod_{n=1}^{\infty} \frac{f\left(\lim_{T \to \infty} \frac{1}{2\pi b_n} \int_{-T}^{T} e^{-ic_n t} [X(t,\omega) - s(t)] dt \right)}{f\left(\lim_{T \to \infty} \frac{1}{2\pi b_n} \int_{-T}^{T} e^{ic_n t} X(t,\omega) dt \right)}$$

a.e. (P).

Other cases where the map Ψ can be determined are the discrete time $S\alpha S$ processes we discussed in Section 2.3, e.g. in the inventible $ARMA(p,q)$ we have

$$\Psi(n,x) = x_n - \sum_{j=1}^{\infty} n_j \, x_{n-j} \, ,$$

so that

$$Z_n(\omega) \;=\; X_n(\omega) - \sum_{j=1}^{\infty} h_j X_{n-j}(\omega),$$

and

$$z_n \;=\; s_n - \sum_{j=1}^{\infty} h_j s_{n-j},$$

where z_n is as in example in Section 2.

It is well known that the likelihood-ratio test is a decision rule that is optimum with respect to either a Neyman-Pearson or Bayes criterion, i.e. if we observe Y and we want to test

$$H_0: Y = X \quad \text{versus} \quad H_A: Y = s+X,$$

the optimal procedure is to reject H_0 if

$$\frac{d\mu_{s+X}}{d\mu_X}\,(Y) \;>\; L$$

for some appropriately determine threshold (dependent on the criterion used).

In the Gaussian case ($\alpha = 2$) this procedures reduces to

$$\sum_{n=1}^{\infty} s_n \, x_n \;>\; L' \, .$$

In the non-Gaussian case, the likelihood ratio is difficult to implement even when it is expressed in the form $\prod_{n=1}^{\infty} f(x_n - s_n)/f(x_n)$.

The log of the ratio $f(x-s)/f(x)$ has been studied in Stuck (1976); using series expansions of the density f. Plots for $\log\{f(x-s)/f(x)\} = l_s(x)$ were presented for some

fixed values of s. Stuck (1976) also investigated the performance of the likelihood (optimal) test versus the linear (Gaussian) decision rule for a finite number of terms and fixed s, i.e. the case where the Radon-Nikodym derivative is given by the finite product

$$\Pi_{n=1}^{N} \frac{f(x_n - s)}{f(x_n)},$$

and the plots showed that the likelihood procedure has in general a much smaller probability of error than the linear rule, even for values of α close to 2, such as $\alpha = 1.95$.

All inventible SαS processes with nonatomic control measure have no (nontrivial) admissible translates when $0 < \alpha < 2$, whereas in the Gaussian case $\alpha = 2$, their set of admissible translates coincides with their RKHS, which is a quite large class of functions. Thus the detection of a nonrandom signal in additive SαS noise satisfying these assumptions can in principle always be achieved with probability one even for signals of comparable smoothness with the noise, i.e. the detection is singular (even though practical detectors achieving this remain to be found). In contrast in the Gaussian case ($\alpha = 2$) the additive signal detection problem is regular for signals with comparable smoothness with the noise (signals in the RKHS), i.e. a Neyman-Pearson test can be constructed having a specified probability of false alarm which maximizes the probability of detection (which is of course always less than 1).

CHAPTER III

RESULTS ON EQUIVALENCE AND SINGULARITY

3.1 On the equivalence and singularity of certain product measures

The Lebesgue decomposition of product measures was first studied and completely solved in Kakutani (1948). His criterion is given in terms of the Hellinger integrals of the marginal measures, which may be difficult to compute, e.g. for stable measures.

The more special problem of translates of product measures with identical one dimensional marginals was settled in Shepp (1965) in the ease of finite Fisher information. It was observed in LeCam (1970) that under LeCam's "Γ" condition the sufficient condition for equivalence in Shepp (1965) can be extended to a more general scenario.

Here we show that under a condition closely related to LeCam's condition, a nearly complete extension of Shepp's theorem holds. As an application the result on equivalence and singularity between a sequence of i.i.d. random variables and an affine transformation of itself extends to a large class of nonGaussian distributions, which includes in particular all stable distributions. Our result also contains that of Steele (1986) on the extension of Shepp's theorem to rigid Euclidean motions (i.e., rotations, translations, and their compositions) of an $\mathbb{R}^k$ vector.

In Section 3.3 these results will be used to study the Lebesgue decomposition between certain SαS processes (e.g., independent increments, harmonizable).

Before stating the main results we need to introduce some concepts for which we refer to Strasser (1985) Chapter 1, Section 2 and Chapter 12 Sections 75 and 78.

3.1.1 Preliminaries

Given two probability measures P and Q on a measurable space $(\Omega, \mathcal{F})$ their normalized Hellinger distance d and integral H are defined by

$$d^2(P,Q) \;=\; \tfrac{1}{2}\textstyle\int_\Omega \mid \left(\tfrac{dP}{d\nu}\right)^{1/2} - \left(\tfrac{dQ}{d\nu}\right)^{1/2}\mid^2 d\nu,$$

$$H(P,Q) \;=\; \int_\Omega \left(\tfrac{dP}{d\nu}\right)^{1/2} \left(\tfrac{dQ}{d\nu}\right)^{1/2} d\nu,$$

where ν is any σ-finite measure dominating $P+Q$, i.e. $P+Q \ll \nu$ (e.g. $\nu = P+Q$). They do not depend on ν and satisfy

$$0 \le H(P,Q) \le 1 \quad\text{and}\quad 1 - H(P,Q) \;=\; d^2(P,Q).$$

Kakutani's theorem states that if $(\mu_n;\, n \in \mathbf{N})$ and $(\lambda_n,\, n \in \mathbf{N})$ are sequences of probability measures and $\mu = \mathbf{X}_{n=1}^\infty \mu_n$ and $\lambda = \mathbf{X}_{n=1}^\infty \lambda_n$ are their product measures, then

$$(3.1.1) \qquad \mu \perp \lambda \quad\Leftrightarrow\quad \Pi_{n=1}^\infty H(\mu_n, \lambda_n) = 0 \quad\Leftrightarrow\quad \sum_{n=1}^\infty d^2(\mu_n, \lambda_n) = \infty$$

and if $\mu_n \sim \lambda_n$ for all n, then

$$(3.1.2) \qquad \mu \sim \lambda \quad\Leftrightarrow\quad \Pi_{n=1}^\infty H(\mu_n, \lambda_n) > 0 \quad\Leftrightarrow\quad \sum_{n=1}^\infty d^2(\mu_n, \lambda_n) < \infty$$

(see Kakutani (1948)).

We consider the following setting. $(\Omega, \mathcal{F}, \nu)$ is a σ-finite measure space, and $\{P_\theta : \theta \in \Theta\}$ a family of probability measures on $(\Omega, \mathcal{F})$ which are absolutely continuous with respect to ν, where Θ is an open subset of $\mathbf{R}^k$. Define $F: \Theta \to L_2(\Omega, \mathcal{F}, \nu) = L_2(\nu)$

by

$$F(\theta) \;=\; 2 \left(\frac{dP_\theta}{d\nu}\right)^{1/2}.$$

F is said to be differentiable at θ, if there exists a map

$$DF(\cdot,\theta) = DF(\theta): \Omega \to \mathbf{R}^k$$

such that

$$\|DF(\theta)\|^2_{L_2(\Omega,\mathcal{F},\nu;\mathbf{R}^k)} \;=\; \int_\Omega \|DF(\omega,\theta)\|^2_{\mathbf{R}^k}\,\nu(d\omega) \;<\; \infty,$$

i.e. $DF(\theta) \in L_2(\Omega,\mathcal{F},\nu;\mathbf{R}^k)$, and

$$\int_\Omega | \, F(\theta+h)-F(\theta)- <DF(\theta),h>_{\mathbf{R}^k} \, |^2 \; d\nu = o(\|h\|^2_{\mathbf{R}^k}) \qquad \text{as } \|h\|_{\mathbf{R}_k} \to 0.$$

As usual F is said to be differentiable (on Θ) if it is differentiable at each $\theta \in \Theta$. The Fisher's information matrix is defined by

$$I(\theta) \;=\; \int_\Omega DF(\theta)\,DF(\theta)^T\,d\nu$$

(where $DF(\theta)^T$ is the transpose of the column vector $DF(\theta)$). It is non negative definite. as $a^T I(\theta)a \;=\; \int_\Omega (a^T DF(\theta))^2 d\nu$, and is positive definite if and only if the components of $DF(\theta)$ are linearly independent functions in $L_2(\nu)$.

3.1.2 Main result

As in LeCam (1970), our purpose is to consider product measures

$$(3.1.3) \qquad \mu = \mathbf{X}_{n=1}^{\infty}\mu_n \quad \text{and} \quad \lambda = \mathbf{X}_{n=1}^{\infty}\lambda_n \; ,$$

where

$$\mu_n = P_\theta \quad \text{and} \quad \lambda_n = P_{\theta+h_n},$$

$\theta \in \Theta$ is fixed and $\theta + h_n \in \Theta$, $n = 1,2,\ldots$ Under the condition

$$"\Gamma": \quad \limsup_{\|h\|_{\mathbf{R}^k} \to 0} H(P_{\theta+h}, P_\theta)/\|h\|_{\mathbf{R}^k} < \infty,$$

LeCam (1970), Proposition 2, proved that $\sum_{n=1}^{\infty}\|h_n\|_{\mathbf{R}^k}^2 < \infty$ implies $\mu \sim \lambda$. Here, under the conditions that $\mathbf{F}$ is differentiable at θ, $I(\theta)$ is positive definite and the probability measures $\{P_\theta; \theta \in \Theta\}$ are sufficiently separated, we obtain necessary and sufficient conditions for equivalence and singularity.

The separation type condition that we assume is

$$(3.1.4) \qquad \text{"for all sufficiently small } \delta > 0, \inf_{\|h\|_{\mathbf{R}^k} > \delta} d^2(P_{\theta+h}, P_\theta) > 0 \text{ "} .$$

Note that if (3.1.4) does not hold then there exists $\delta > 0$ and a sequence $(h_n; n \in \mathbf{N})$ in $\mathbf{R}^k$ with $\|h_n\|_{\mathbf{R}^k} > \delta$ such that $d^2(P_{\theta+h_n}, P_\theta) \to 0$ as $n \to \infty$, i.e. for any $\epsilon > 0$ and for n large, $d(P_{\theta+h_n}, P_\theta) < \epsilon$, and using a well known relation between Hellinger distance and total variation distance (see e.g. Strasser (1985)) we have

$$d_v(P_{\theta+h_n}, P_\theta) = \sup_{E \in \mathcal{F}} | P_{\theta+h_n}(E) - P_\theta(E)| \leq 2^{1/2} d(P_{\theta+h_n}, P_\theta) < 2^{1/2}\epsilon.$$

Hence if condition (3.1.4) is violated, for ϵ arbitrarily small there exists

$h \in \mathbf{R}^k$ with $\|h\|_{\mathbf{R}^k} < \delta$ such that

$$\sup_{E \in \mathcal{F}} | P_{\theta+h}(E) - P_\theta(E) | < \epsilon.$$

Thus in this sense (3.1.4) is a separation type condition.

We should also mention that the proof of the sufficient condition for equivalence can be obtained directly from LeCam (1970), Proposition 2, since L_2-differentiability is clearly stronger than condition "Γ", but we choose to include a complete simple proof here.

<u>Proposition</u> 3.1.1. Let μ and λ be as in (3.1.3), F be differentiable at $\theta \in \Theta$ and $I(\theta)$ be positive definite.

i)   If $0 < \|h_n\|_{\mathbf{R}^k} \to 0$ as $n \to \infty$, then

$$\mu \perp \lambda \quad \Leftrightarrow \quad \sum_{n=1}^\infty \|h_n\|_{\mathbf{R}^k}^2 = \infty$$

and

$$\mu \sim \lambda \quad \Leftrightarrow \quad \sum_{n=1}^\infty \|h_n\|_{\mathbf{R}^k}^2 < \infty.$$

ii)   If condition (3.1.4) is satisfied then i) holds for any sequence $(h_n; n \in \mathbf{N})$.

<u>Proof.</u> If since F is differentiable at θ, as $0 < \|h\|_{\mathbf{R}^k} \to 0$ we have

$$\left\| \frac{F(\theta+h) - (F(\theta)}{\|h\|_{\mathbf{R}^k}} - \frac{<DF(\theta), h>_{\mathbf{R}^k}}{\|h\|_{\mathbf{R}^k}} \right\|_{L_2(\nu)} = o(1) ,$$

hence

$$\left| \frac{\|F(\theta+h) - F(\theta)\|_{L_2(\nu)}}{\|h\|_{\mathbf{R}^k}} - \frac{\|<DF(\theta),h>_{\mathbf{R}^k}\|_{L_2(\nu)}}{\|h\|_{\mathbf{R}^k}} \right| = o(1) .$$

Thus for any $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that if $0 < \|h\|_{\mathbf{R}^k} < \delta$,

$$\frac{\| <DF(\theta),h>_{\mathbf{R}^k} \|_{L_2(\nu)}}{\|h\|_{\mathbf{R}^k}} - \epsilon < \frac{\|F(\theta+h)-F(\theta)\|_{L_2(\nu)}}{\|h\|_{\mathbf{R}^k}} < \frac{\| <DF(\theta),h>_{\mathbf{R}^k} \|_{L_2(\nu)}}{\|h\|_{\mathbf{R}^k}} + \epsilon.$$

Now

$$\| <DF(\theta),h>_{\mathbf{R}^k} \|_{L_2(\nu)}^2 = \int_\Omega | <DF(\theta),h>_{\mathbf{R}^k} |^2 d\nu = h^\mathsf{T} I(\theta)h,$$

hence for all $h \in \mathbf{R}^k$, $h \neq 0$,

$$k(\theta) \leq \frac{\| <DF(\theta),h>_{\mathbf{R}^k} \|_{L_2(\nu)}}{\|h\|_{\mathbf{R}^k}} \leq K(\theta)$$

where $k(\theta)$ and $K(\theta)$ are the smallest and the largest eigenvalues of $I(\theta)$. Since $I(\theta)$ is positive definite $k(\theta) > 0$ and we can choose $\epsilon > 0$ with $0 < k(\theta) - \epsilon$ so that for all $0 < \|h\|_{\mathbf{R}^k} < \delta$,

$$0 < L(\theta) < \frac{\|F(\theta+h) - F(\theta)\|_{L_2(\nu)}}{\|h\|_{\mathbf{R}^k}} < U(\theta)$$

where $L(\theta) = k(\theta) - \epsilon$ and $U(\theta) = K(\theta) + \epsilon$. Thus since $d^2(P_\theta, P_{\theta'}) = \|F(\theta) - F(\theta')\|_{L_2(\nu)}^2 / 8$ we have for n large

$$0 < \frac{L^2(\theta) \|h_n\|_{\mathbf{R}_k}^2}{8} < d^2(\mu_n, \lambda_n) < \frac{U^2(\theta) \|h_n\|_{\mathbf{R}^k}^2}{8}$$

and the result follows from (3.1.1.) and (3.1.2).

   ii)   If (3.1.4) is satisfied and $\|h_n\|_{\mathbf{R}^k} \nrightarrow 0$, then there exist $\delta > 0$ and a

subsequence $(n_j; j \in \mathbf{N})$ with $\|h_{n_j}\|_{\mathbf{R}^k} > \delta$ such that

$$\sum_{n=1}^{\infty} d^2 (\mu_n, \lambda_n) \geq \sum_{j=1}^{\infty} d^2 (\mu_{n_j}, \lambda_{n_j}) \geq$$

$$\geq \sum_{j=1}^{\infty} \inf d^2 (P_{\theta+h}, P_\theta) = \infty \ ,$$

and from (3.1.1), $\mu \perp \lambda$. This combined with i) gives the result   □

It should be mentioned that the differentiability of $F(\theta)$ is generally difficult to verify, but it can be shown that it is implied by the classical regularity conditions, usually called Cramer-Wald and Hajek's conditions, which play an important role in statistical estimation theory and are in principle easy to check (see e.g. Strasser (1985), §77). However, L_2-differentiability is weaker than any of these classical conditions, and the definition of Fisher information presented here extends the classifical one. namely

$$I(\theta) = - E\left(\frac{\partial^2}{\partial \theta^2} \ln \left(\frac{dP_\theta}{d\nu} \right) \right)$$

under the usual conditions on $\dfrac{dP_\theta}{d\nu}$.

3.1.3 Examples

3.1.3.1 Affine Transformation in $\mathbf{R}^k$.

Suppose $(X_n; n \in \mathbf{N})$ is a sequence of i.i.d. random vectors in $\mathbf{R}^k$, $(A_n; n \in \mathbf{N})$ a sequence of $k \times k$ matrices and $(b_n; n \in \mathbf{N})$ a sequence of vectors. If we want to compare the sequence of random vectors $(X_n; n \in \mathbf{N})$ with $(A_n X_n + b_n; n \in \mathbf{N})$ we can take as parameter space Θ any open subset of

$$\{\theta = (A,b); A = (a_{ij}): k \times k \text{ matrix}, b = (b_i) \in \mathbf{R}^k\} \equiv$$

$$\equiv \{\theta; \theta = (a_{11},...,a_{1k},...,a_{kk},b_1,...,b_k)\} \equiv$$

$$\equiv \mathbf{R}^{k^2+k} \equiv \mathbf{R}^{k^2} \times \mathbf{R}^k$$

containing the point $(I,0)$, with

$$\|A\|^2_{\mathbf{R}^{k \times k}} = \sum_{i,j=1}^{k} a_{ij}^2, \quad \|b\|^2_{\mathbf{R}^k} = \sum_{i=1}^{k} b_i^2,$$

and

$$\|\theta\|^2_{\mathbf{R}^{(k \times k)+k}} = \|A\|^2_{\mathbf{R}^{k \times k}} + \|b\|^2_{\mathbf{R}^k}.$$

With P the common distribution of the i.i.d. random vectors X_n and $\theta = (A,b)$ we define

$$(3.1.5) \qquad P_\theta(B) = P_{(A,b)}(B) = P(\{AX_n + b \in B\})$$

and note that $P = P_{(I,0)}$. From proposition 3.1.1 we have the following

Corollary 3.1.2. Let the probability measures P_θ defined as in (3.1.5) be such that for an open set $\Theta \subset \mathbf{R}^{k^2+k} \equiv \mathbf{R}^{k^2} \times \mathbf{R}^k$ with $(I,0) \in \Theta$, the family $\{P_\theta; \theta \in \Theta\}$ is dominated by ν, $F(\theta)$ is differentiable at $(I,0)$ and $I(I,0)$ is positive definite. If $A_n \to I$ and $b_n \to 0$ as $n \to \infty$ then

$$(X_n) \sim (A_n X_n + b_n) \quad \Leftrightarrow \quad \sum_{n=1}^{\infty} \|b_n\|^2_{\mathbf{R}^k} < \infty \quad \text{and}$$

$$\sum_{n=1}^{\infty} \|I - A_n\|_{\mathbf{R}^{k \times k}}^2 < \infty,$$

$$(X_n) \perp (A_n X_n + b_n) \quad \Leftrightarrow \quad \sum_{n=1}^{\infty} \|b_n\|_{\mathbf{R}^k}^2 = \infty \quad \text{or}$$

$$\sum_{n=1}^{\infty} \|I - A_n\|_{\mathbf{R}^{k \times k}}^2 = \infty.$$

Furthermore if $\{P_\theta; \theta \in \Theta\}$ satisfies condition (3.1.4) the above conclusions hold for all sequence (A_n, b_n) in Θ.

<u>Proof</u>. Putting $\theta = (I, 0)$ and $(A_n, b_n) = \theta + h_n$ we have $h_n = (A_n - I, b_n)$ and

$$\sum_{n=1}^{\infty} \|h_n\|_{\mathbf{R}^{(k \times k) + k}}^2 = \sum_{n=1}^{\infty} \|A_n - I\|_{\mathbf{R}^{k \times k}}^2 + \sum_{n=1}^{\infty} \|b_n\|_{\mathbf{R}^k}^2.$$

The conclusion then follows from Proposition 3.1.1. $\qquad\qquad\qquad\qquad$ □

<u>Remarks</u>: a) Since the space of $k \times k$ matrices is finite dimensional, any norm can be used in place of $\|\cdot\|_{\mathbf{R}^{k \times k}}$.

b) When $A_n = I$ for all n, Corollary 3.1.2 extends the result in Shepp (1965) on translates from sequences of random variables (in $\mathbf{R}^1$) to sequences of random vectors (in $\mathbf{R}^k$).

c) Corollary 3.1.2 contains the results in Steele (1986), Theorems 1 and 2, who considers the case where A_n is a rotation, i.e. $A_n x + b_n$ is a rigid motion of $x \in \mathbf{R}^k$.

d) When the X_n's are Gaussian random variables with mean zero and variance one, the result of Corollary 3.1.1 can be checked directly through the computation of the Hellinger integrals $H(P_{(1,0)}, P_{(a_n, b_n)})$. However, even for Gaussian random vectors the computation of Hellinger integrals is not simple in higher

dimensions.

3.1.3.2 Stable Sequences.

Here we consider general (skewed, not necessarily symmetric) stable densities. We denote by $f_{(\alpha,\beta,a,b)}$ the general univariate stable density whose characteristics function is of the form

$$\int_{-\infty}^{\infty} \exp(iux) f_{(\alpha,\beta,a,b)}(x) dx = \begin{cases} \exp\{-|au|^{\alpha}\exp[-i\pi\beta \, \text{sgn}(u)/2]+ibu\}, & \text{if } \alpha \neq 1 \\ \\ \exp\{-|au|-i(2\beta/\pi)au \, \ln(|au|)+ibu\}, & \text{if } \alpha = 1 \end{cases}$$

where $0 < \alpha \leq 2$, $|\beta| \leq \alpha \wedge (2-\alpha)$, $a > 0$ and $-\infty < b < \infty$.

Corollary 3.1.3. Let $(X_n; n \in \mathbb{N})$ be a sequence of i.i.d. stable random variables with density $f(\alpha_0, \beta_0, a_0, b_0)$ and let $(Y_n; n \in \mathbb{N})$ be a sequence of independent stable random variables where the density of each Y_n is $f(\alpha_n, \beta_n, a_n, b_n)$ with $(\alpha_n, \beta_n, a_n, b_n) \to (\alpha_0, \beta_0, a_0, b_0)$. Then

$$(X_n) \sim (Y_n) \Leftrightarrow \begin{cases} \sum_{n=1}^{\infty} (\alpha_n - \alpha_0)^2 < \infty, \; \sum_{n=1}^{\infty} (\beta_n - \beta_0)^2 < \infty, \\ \\ \sum_{n=1}^{\infty} (a_n - a_0)^2 < \infty, \text{ and } \sum_{n=1}^{\infty} (b_n - b_0)^2 < \infty, \end{cases}$$

and

$$(X_n) \perp (Y_n) \Leftrightarrow \begin{cases} \sum_{n=1}^{\infty} (\alpha_n - \alpha_0)^2 = \infty, \text{ or } \sum_{n=1}^{\infty} (\beta_n - \beta_0)^2 = \infty, \text{ or} \\ \\ \sum_{n=1}^{\infty} (a_n - a_0)^2 = \infty, \text{ or } \sum_{n=1}^{\infty} (b_n - b_0)^2 = \infty. \end{cases}$$

Proof: Let Θ be any open subset of

$$\{\theta = (\alpha,\beta,a,b); \; \alpha \in (0,1)\cup(1,2), \; |\beta| < \alpha \wedge (2-\alpha), \; a > 0, \; -\infty < b < \infty\}$$

containing the point $\theta_0 = (\alpha_0, \beta_0, a_0 b_0)$. It is known that the densities $\{f_\theta; \theta \in \Theta\}$ satisfy the usual regularity conditions of Cramer-Wald (DuMouchel (1973), page 952); hence $f_\theta^{1/2}$, $\theta \in \Theta$, are L_2(Leb)-differentiable (see e.g. Strasser (1985), §77). Moreover the Fisher information matrix $I(\theta_0)$ is positive definite (DuMouchel (1973), page 954). Therefore the assumptions of Proposition 3.1.1 hold at $\theta_0 = (\alpha_0, \beta_0, a_0, b_0)$. Hence for $h_n = (\alpha_n - \alpha_0, \beta_n - \beta_0, a_n - a_0, b_n - b_0)$ we have

$$\|h_n\|_{\mathbf{R}^4}^2 = (\alpha_n - \alpha_0)^2 + (\beta_n - \beta_0)^2 + (a_n - a_0)^2 + (b_n - b_0)^2$$

and the result follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\Box$

Corollary 3.1.4. Let $(X_n; n \in \mathbf{N})$ be a sequence of i.i.d. stable random variables with density $f_{(\alpha,0,1,0)}$ and $\alpha \in (0,1) \cup (1,2)$, i.e. X_n are standard $S\alpha S$ random variables, and let (a_n, b_n) and (a'_n, b'_n) be two sequences in $(-\infty, 0) \cup (0, \infty) \times (-\infty, \infty)$ with $a_n/a'_n \to 1$ and $(b_n - b'_n)/a'_n \to 0$. Then

$$(a_n Y_n + b_n) \sim (a'_n Y_n + b'_n) \quad \Leftrightarrow \quad (X_n) \sim \left(\frac{a_n}{a'_n} X_n + \frac{b_n - b'_n}{a'_n}\right)$$

$$\Leftrightarrow \quad \sum_{n=1}^{\infty} \left(1 - \frac{|a_n|}{|a'_n|}\right)^2 < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \left(\frac{b_n - b'_n}{a'_n}\right)^2 < \infty,$$

and

$$(a_n Y_n + b_n) \perp (a'_n Y_n + b'_n) \quad \Leftrightarrow \quad (X_n) \perp \left(\frac{a_n}{a'_n} Y_n + \frac{b_n - b'_n}{a'_n}\right)$$

$$\Leftrightarrow \quad \sum_{n=1}^{\infty} \left(1 - \frac{|a_n|}{|a'_n|}\right)^2 = \infty \quad \text{or} \quad \sum_{n=1}^{\infty} \left(\frac{b_n - b'_n}{a'_n}\right)^2 = \infty.$$

Proof: The equivalence of i) and ii) follows since the map $(x_n) \to ((x_n - b'_n)/a'_n)$ is invertible. The equivalence of ii) and iii) follows from Corollary 3.1.1. (putting as parameters $\alpha_n = \alpha$, $\beta_n = 0$, $a_0 = 1$ and $b_0 = 0$, $\frac{a_n}{a'_n}$, $\frac{b_n - b'_n}{a'_n}$; $n = 1, 2, ...$) for

$\alpha \in (0,1) \cup (1,2).$ □

The separation condition (3.1.4) for the case where all the parameters but

translation (a) are kept fixed in a stable probability density follows from the inequality

in Ibragimov and Has'minski (1981), Example 3, Page 57; and when $\beta=0$ and α is fixed

it has been proved by Kanter (1975). Condition (3.1.4) is not known to be true for the

case where all the parameters vary jointly. Hence singularity does not follow for all

sequences $h_n = (\alpha_n, \beta_n, a_n, b_n)$, n=1,2,... with $\sum_{n=1}^{\infty} \|h_n\|_{\mathbf{R}^k}^2 = \infty$ (cf Proposition

3.1.1).

In the next proposition we explore the tail behavior of a stable distribution to show

that two infinite sequences of independent symmetric stable random variables with

different indexes of stability are singular.

<u>Proposition 3.1.5</u> Let $X=(X_n; n \in \mathbf{N})$ and $Y=(Y_n; n \in \mathbf{N})$ be two sequences of

independent (nondegenerate) symmetric stable random variables with indexes of

stability α and β respectively. If $\alpha \neq \beta$ then μ_X and μ_Y are singular.

<u>Proof</u>: Assume $\alpha < \beta$. For each $\gamma \in (0,2)$ let Z_γ denote a SγS r.v. with $\|Z_\gamma\|_\gamma = 1$.

Thus

$$\mu_n(B) \triangleq P(X_n \in B) = P(\|X_n\|_\alpha Z_\alpha \in B),$$

$$\lambda_n(B) \triangleq P(Y_n \in B) = P(\|Y_n\|_\beta Z_\beta \in B).$$

Since

$$a^\gamma P(|Z_\gamma| > a) \;\longrightarrow\; C_\gamma \qquad\qquad \text{as } a \to \infty$$

(where C_γ is a positive constant), given any $\epsilon > 0$, there exist $M_{\gamma,\epsilon}$ such that for $a > M_{\gamma,\epsilon}$,

$$\frac{C_\gamma - \epsilon}{a^\gamma} < P(|Z_\gamma| > a) < \frac{C_\gamma + \epsilon}{a^\gamma}$$

(see e.g. Feller (1966)).

From now on fix ϵ such that $0 < \epsilon < \min(C_\alpha, C_\beta)$.

Case 1. Assume

$$\sigma_n \triangleq \frac{\|X_n\|_\alpha}{\|Y_n\|_\beta} \rightarrow 0, \qquad\qquad\qquad \text{as } n \rightarrow \infty.$$

Define $\Psi: X^N \rightarrow X^N$ by

$$\Psi(x) = (\Psi_n(x) = x_n / \|Y_n\|_\beta;\ n \in N).$$

It follows that Ψ is an i.i.d. sequence of standard $S\beta S$ r.v.'s under μ_Y and under μ_X an independent sequence of $S\alpha S$ r.v.'s with

$$\|\Psi_n\|_\alpha = \|X_n\|_\alpha / \|Y_n\|_\beta = \sigma_n.$$

As before let d_V denote the total variation distance between probability measures. For $a > \max(M, M \sup_n \sigma_n)$ we have

$$d_V(\lambda_n \Psi_n^{-1}, \mu_n \Psi_n^{-1}) \geq P(|Z_\beta| > a) - (|\sigma_n Z_\alpha| > a)$$

$$> \frac{C_\beta - \epsilon}{a^\beta} - \sigma_n^\alpha \frac{C_\alpha + \epsilon}{a^\alpha}.$$

Thus

$$\liminf_{n \to \infty} d_v \left(\lambda_n \Psi_n^{-1}, \mu_n \Psi_n^{-1}\right) \geq \frac{C_\beta^-}{a^\beta} > 0.$$

Since $d_v (\cdot, \cdot) \leq \sqrt{2} \, d (\cdot, \cdot)$ where d denotes the Hellinger distance (see e.g. Strasser (1985)) we have

$$\sum_{n=1}^{\infty} d \left(\mu_n \Psi_n^{-1}, \lambda_n \Psi_n^{-1}\right) = \infty$$

and therefore by Kakutani's Theorem (see Page 47) $\mu_X \Psi^{-1} \perp \mu_Y \Psi^{-1}$, which implies $\mu_X \perp \mu_Y$.

Case 2. Assume $\sigma_n \not\to 0$. Thus there exist $\delta > 0$ and a sequence $(n_k; k \in \mathbf{N})$ such that $\sigma_{n_k} \geq \delta$, i.e. $\sigma_{n_k}^{-1} \leq \delta^{-1}$.

Define $\Phi: \mathbf{X}^{\mathbf{N}} \to \mathbf{X}^{\mathbf{N}}$ by

$$\Phi(x) = (\Phi_k(x) = x_k / \|X_{n_k}\|_\alpha; \, k \in \mathbf{N}).$$

Thus Φ is an i.i.d. sequence of standard SαS r.v.'s under μ_X and under μ_Y an independent sequence of SβS r.v.'s with

$$\|\Phi_k\|_\beta = \|Y_{n_k}\|_\beta / \|X_{n_k}\|_\alpha = \sigma_{n_k}^{-1}.$$

For $a > \max (M, \delta^{-1}M) \geq \max (M, \sigma_{n_k}^{-1}M)$ we have

$$d_v(\mu_k \Phi_k^{-1}, \lambda_k \Phi_k^{-1}) \geq P(|Z_\alpha| > a) - P(|\sigma_{n_k}^{-1} Z_\beta| > a)$$

$$> \frac{C_{\alpha-\epsilon}}{a^{\alpha}} - \sigma_{n_k}^{-1} \frac{C_{\beta+\epsilon}}{a^{\beta}}$$

$$> \frac{C_{\alpha-\epsilon}}{a^{\alpha}} - \delta^{-1} \frac{C_{\beta+\epsilon}}{a^{\beta}} \triangleq \delta'(a).$$

Since $\alpha < \beta$, we have $\delta'(a) > 0$ if and only if $a^{\beta-\alpha} > \delta^{-1}(C_{\beta}+\epsilon)/(C_{\alpha}-\epsilon)$. Thus, fixing $a > \max \{M, \delta^{-1}M, [M(C_{\beta} + \epsilon)/(C_{\alpha} - \epsilon)]^{1/(\beta-\alpha)}\}$ we obtain

$$\limsup_{n \to \infty} d_v(\mu_n \Phi_n^{-1}, \lambda_n \Phi_n^{-1}) > \delta'(a) > 0$$

and the conclusion follows as in case 1. $\qquad\qquad\qquad\qquad\qquad\qquad$ □

Remark: If $\beta = 2$, Proposition 3.1.5 remains true with minor modifications in its proof.

3.2. Remarks on singularity and absolute continuity of pth-order and SαS processes

A necessary condition for equivalence of two Gaussian processes is the setwise equality of their RKHS (or the equivalence of their RKHS norms). We show that this result remains true for SαS processes with the function space $\mathbb{F}$ replacing the RKHS's. Further for p^{th} order processes with $1 < p < 2$, a necessary condition for abolute continuity and a sufficient condition for singularity are presented analogous to those of Fortet (1973) for second order processes.

Let $X_i = (X_i(t); t \in \mathbb{T})$, $i = 1,2$, be two p^{th} order processes. We say that X_1 dominates X_2 if there exists $0 < K < \infty$ such that for all $N \in \mathbb{N}$, $a_1,...,a_N \in \mathbb{N}$ and $t_1,...,t_N \in \mathbb{T}$,

$$\|\sum_{n=1}^{N} a_n X_2(t_n)\|_{L_p(P)} \leq K \|\sum_{n=1}^{N} a_n X_1(t_n)\|_{L_p(P)}.$$

Proposition 3.2.1. Let $\mathbb{F}_i = \mathbb{F}(X_i)$, $i = 1,2$.

i)   If X_1 dominates X_2, then $\mathbf{F}_2 \subset \mathbf{F}_1$.

ii) X_1 dominates X_2 if and only if there exists a bounded linear transformation $\Theta: \mathcal{L}(X_1) \rightarrow \mathcal{L}(X_2)$, satisfying $\Theta(X_1(t)) = X_2(t)$, $t \in \mathbf{T}$. Consequently, if X_1 dominates X_2 and vice verse, then $\mathbf{F}_1 = \mathbf{F}_2$ (setwise), $\|\cdot\|_{\mathbf{F}_1}$ and $\|\cdot\|_{\mathbf{F}_2}$ are equivalent, and the transformation Θ has bounded invense.

Proof. i)   Suppose X_1 dominates X_2. Then for all functions s,

$$\frac{1}{K} \frac{|\sum_{n=1}^{N} a_n s(t_n)|}{\|\sum_{n=1}^{N} a_n X_1(t_n)\|_{L_p(P)}} \leq \frac{|\sum_{n=1}^{N} a_n s(t_n)|}{\|\sum_{n=1}^{N} a_n X_2(t_n)\|_{L_p(P)}}$$

and by taking supremum over N, $a_1,\dots,a_N$, $t_1,\dots,t_N$,

$$\frac{1}{K} \|s\|_{\mathbf{F}_1} \leq \|s\|_{\mathbf{F}_2}.$$

Thus if $s \in \mathbf{F}_2$, it follows that $s \in \mathbf{F}_1$, which proves i).

ii)   Let $\Theta: \mathcal{l}(X_1) \rightarrow \mathcal{L}(X_2)$ be defined by

$$\Theta(\sum_{n=1}^{N} a_n X_1(t_n)) = \sum_{n=1}^{N} a_n X_2(t_n).$$

It is clear that Θ is a well defined bounded linear transformation and as such it can be extended to $\mathcal{L}(X_1)$ if and only if X_1 dominates X_2.   □

For SαS processes, the next Proposition shows that mutual domination is a necessary condition for absolute continuity, i.e. non domination is a sufficient condition for singularity. This Proposition is a stochastic process version of Proposition 7 in Zinn (1975).

<u>Proposition 3.2.2</u>. Let $X_i = (X_i(t); t \in \mathbf{T})$, $i = 1,2$, be two SαS processes. If μ_{X_1} and μ_{X_2} are not singular, then X_1 dominates X_2, X_2 dominates X_1, and $\mathbf{F}_1 = \mathbf{F}_2$. Equivalently if $\mathbf{F}_1 \neq \mathbf{F}_2$ then either X_1 does not dominate X_2 or X_2 does not dominate X_1 and $\mu_{X_1} \perp \mu_{X_2}$.

<u>Proof</u>: Since $\|\cdot\|_{L_p(P)} = C_{p,\alpha_i}\|\cdot\|_{\alpha_i}$, X_1 dominates X_2 if and only if

$$\|\sum_{n=1}^N a_n X_2(t_n)\|_{\alpha_2} \leq K\|\sum_{n=1}^N a_n X_1(t_n)\|_{\alpha_1}.$$

Assume X_1 does not dominate X_2. Then for any positive sequence $K_n \to \infty$, as $n \to \infty$, there exist

$$Y_n^{(i)} = \sum_{k=1}^{N_n} a_{n,k} X_i(t_{n,k}), \quad i = 1,2,...$$

such that

$$\|Y_n^{(2)}\|_{\alpha_2} \geq K_n \|Y_n^{(1)}\|_{\alpha_1}, \quad n = 1,2,....$$

Without loss of generality we can assume $\|Y_n^{(2)}\|_{\alpha_2} = 1$ for all n. Thus

$$\|Y_n^{(1)}\|_{\alpha_2} \leq \frac{1}{K_n} \to 0 \qquad\qquad \text{as } n \to \infty.$$

Now consider the sequence of random variables $(Y_n; n \in \mathbf{N})$ defined on $(\mathbf{X}^{\mathbf{T}}, \mathbf{C})$ by

$$Y_n(x) = \sum_{k=1}^{N_n} a_{n,k} \, x(t_{n,k}), \quad x \in \mathbf{X}^{\mathbf{T}}.$$

It follows that

$$\int_{\mathbf{X}^{\mathbf{T}}} \exp(iuY_n) \, d\mu_{X_1} = \exp(-\|Y_n^{(1)}\|_{\alpha_1}^{\alpha_1} \mid u\mid^{\alpha_1}) \to 1$$

as $n \to \infty$. Hence a subsequence $(Y_{n_k}; k \in \mathbb{N})$ can be chosen such that if $C_0 = \{x; Y_{n_k}(x) \to 0, \text{ as } k \to \infty\}$, then $\mu_{X_1}(C_0) = 1$. Clearly C_0 is a measurable linear subspace of $\mathbf{X}^{\mathbb{T}}$ and, since μ_{X_2} is a $S\alpha_2 S$ measure on $(\mathbf{X}^{\mathbb{T}}, \mathbb{C})$, it follows by the zero-one law for stable measures (Dudley and Kanter (1974)) that $\mu_2(C_0) = 0$ or 1. On the other hand,

$$\int_{\mathbf{X}^{\mathbb{T}}} \exp(iuY_{n_k}) \, d\mu_{X_2} = \exp(-\|Y_{n_k}\|_{\alpha_2}^{\alpha_2} |u|^{\alpha_2}) = \exp(-|u|^{\alpha_2})$$

which implies that $\mu_2(C_0) = 0$ and thus $\mu_{X_1} \perp \mu_{X_2}$. □

The crucial result used in the proof of Proposition 3.2.2 is the zero-one law. This result is not available for general p^{th} order processes but the proposition has some partial analogs for certain p^{th} order processes.

As in Fortet (1973) we call a pth order process $X = (X(t); t \in \mathbb{T})$ non-reduced if there exists some $\epsilon \in (0,1]$ such that for all countable subsets $\mathbb{T}_0$ of $\mathbb{T}$, $P(\{\omega; x(t,\omega) = 0, t \in \mathbb{T}_0\}) \geq \epsilon$; otherwise X is called reduced. Nontrivial $S\alpha S$ processes are reduced. When X is separable and $\mathbb{T}$ an interval of the real line Fortet (1973) showed that X is reduced if and only if $P(\{X(t) = 0, t \in \mathbb{T}\}) = 0$ and nonreduced if and only if $P(X(t) = 0, t \in \mathbb{T}) \geq \epsilon$ for some $\epsilon \in (0,1]$.

Next we generalize to p^{th} processes with $1 < p < 2$ the results in Fortet (1973), Théorèmes (3.2) and (3.3.2). The proof is identical to Fortet's and is presented in a shorter form.

Proposition 3.2.3. Let $X_i = (X_i(t); t \in \mathbb{T})$. $i = 1,2$, be p^{th} order processes with $1 < p < 2$ and $\mathbb{F}_i = \mathbb{F}(X_i)$

    i)   If $\mu_{X_2} \ll \mu_{X_1}$ then $\mathbb{F}_1 \cap \mathbb{F}_2$ is dense in $\mathbb{F}_2$.

ii)   If either X_1 or X_2 is reduced, and $\mathbb{F}_1 \cap \mathbb{F}_2 = \{0\}$, then $\mu_{X_1} \perp \mu_{X_2}$.

<u>Proof</u>. i) Fix $s \in \mathbb{F}_2$. By Proposition 2.1.2 we have

$$s(t) = E(X_2(t)Y^{<p-1>}) = \int_{X^T} x(t)\, a(x)^{<p-1>} \mu_{X_2}(dx)$$

where $Y \in \mathcal{L}(X_2)$ and $a(x)$ is a representation of Y in $L_p(\mu_{X_1})\text{-}\overline{sp}\{x(t);\, t \in \mathbb{T}\} \subset X^{\mathbb{T}}$, $Y(\omega) = a(X(\cdot,\omega))$. Let

$$\mu_{X_2}(E) = \int_E g\, d\mu_{X_1} + \mu_{X_2}(E \cap N)$$

be the Lebesgue decomposition of μ_{X_2} with respect to μ_{X_1}. Define $E_n = \{x: 0 < g(x) \le n\} \cap N^c$ and

$$\begin{aligned}
s_n(t) &= \int_{X^T} x(t)\, a(x)^{<p-1>} 1_{E_n}(x)\, \mu_{X_2}(dx) \\
&= \int_{X^T} x(t)\, a(x)^{<p-1>} g(x) 1_{E_n}(x)\, \mu_{X_1}(dx).
\end{aligned}$$

Since $a^{<p-1>} 1_{E_n} \in L_{p*}(\mu_{X_2})$ and $a^{<p-1>} g 1_{E_n} \in L_{p*}(\mu_{X_1})$ we have $s_n \in \mathbb{F}_1 \cap \mathbb{F}_2$. Also

$$|\textstyle\sum_{k=1}^K c_k(s-s_n)(t_k)| = |\int_{X^T} \sum_{k=1}^K c_k\, x(t_k) a(x)^{<p-1>} 1_{E_n^c}(x) \mu_{X_2}(dx)|$$

$$\le [\int_{X^T} |\textstyle\sum_{k=1}^K c_k\, x(t_k)|^p \mu_{X_2}(dx)]^{1/p} [\int_{X^T} |\, a^{<p-1>}|^{p^*} 1_{E_n^c} d\mu_{X_2}]^{1/p^*}.$$

Thus

$$\|s - s_n\|_{F_2}^{p^*} \leq \int_{E_n^c} |a^{<p-1>}|^{p^*} d\mu_{X_2} = \int_{\{g \geq n\}} |a|^p g d\mu_{X_1} \quad \rightarrow \quad 0$$

as $n \rightarrow \infty$, i.e. $F_1 \cap F_2$ is dense in F_2,

    ii)   For a fixed $t_0 \in T$, let $a_0(x) = x(t_0)$ and define

$$s^{(0)}(t) = \int_{X^T} x(t) a_0(x)^{<p-1>} \mu_{X_2}(dx).$$

By Proposition 2.1.2., $s^{(0)} \in F_2$, since $a_0(x) \in L_{p*}(\mu_{X_2})$. Let

$$s_n^{(0)}(t) = \int_{X^T} x(t) a_0(x)^{<p-1>} 1_{E_n}(x) \mu_{X_2}(dx)$$

$$= \int_{X^T} x(t) a_0(x)^{<p-1>} g(x) 1_{E_n}(x) \mu_{X_1}(dx)$$

so that $s_n^{(0)} \in F_1 \cap F_2$. Since $F_1 \cap F_2 = \{0\}$, $s_n^{(0)} \equiv 0$, i.e. $s_n^{(0)}(t) = 0$ for all $t \in T$. In particular

$$s_n^{(0)}(t_0) = \int_{\{0 < g < n\}} |x(t_0)|^p g(x) \mu_{X_1}(dx) = 0 \quad \text{for } n = 1,2,...,$$

and hence

$$\int_{\{0 < g < \infty\}} |x(t_0)|^p g(x) \mu_{X_1}(dx) = 0.$$

Consequently, since $t_0 \in T$ is arbitrary, we have $x(t) = 0$ a.e. (μ_{X_1}) on $\{0 < g < \infty\}$ for each $t \in T$. But this implies that X_1 is non-reduced if

$$\mu_{X_1}(\{x; x(t) = 0, t \in T\}) \geq \mu_{X_1}(\{x; 0 < g(x) < \infty\}).$$

On the other hand if $\mu_{X_1}(\{x; 0 < g(x) < \infty\}) > 0$ then $x(t) = 0$ a.e. $(g\mu_{X_1})$ for each t and $\int\limits_{[0<g<\infty]} g d\mu_{X_1} > 0$. Hence

$$\mu_{X_2}(\{x; x(t) = 0, t \in T_0\}) \geq \int\limits_{[0<g<\infty]} g d\mu_{X_1} > 0,$$

i.e. X_2 is nonreduced. Since either X_1 or X_2 is reduced we must have

$\mu_{X_1}(\{x; 0 < g(x) < \infty\}) = 0$, i.e. $\mu_{X_1} \perp \mu_{X_2}$. $\qquad\qquad\Box$

3.3 Dichotomies for certain SαS processes

In the study of Lebesgue decomposition of probability measures in infinite dimensional spaces the following dichotomy "two measures are either mutually absolutely continuous or singular", has been proved for product measures (Kakutani (1948)), for Gaussian measures (Feldman (1958) and Hajek (1958)), certain ergodic measures (Kanter (1977)), etc. In Section 2, Chapter II we showed that this dichotomy prevails for admissible translates of certain SαS processes. A dichotomy for general SαS measures has been conjectured by Chatterji and Ramaswamy (1982) but the problem seems to remain open.

In this section we show that a dichotomy holds for certain SαS processes, e.g. independently scattered SαS random measures and harmonizable SαS processes. Necessary and sufficient conditions for equivalence and singularity are given and hold for all $\alpha \in (0,2)$.

We continue to use the notation introduced in Section 2, Chapter II. Through this section we make the assumption that the control measures are not purely atomic with a finite number of atoms. This is equivalent to the infinite dimensionality of the linear space of the processes. When they are finite dimensional we always have equivalence since stable densities are everywhere positive. We start by proving a dichotomy for

independently scattered symmetric stable random measures.

Proposition 3.3.1. For i=1,2, let $Z_i = (Z_i(B); B \in \mathfrak{I})$ be an independently scattered symmetric stable random measure with index of stability $\alpha_i \in (0,2)$ and control measure m_i which is not purely discrete with a finite number of atoms. Then μ_{Z_1} and μ_{Z_2} are mutually absolute continuous if and only if the following conditions are satisfied

    i) $\alpha_1 = \alpha_2 = \alpha$,

    ii) $m_{1,d} = m_{2,d}$,

    iii) m_1 and m_2 have the same set of atoms $A = \{a_n; n \in \mathbf{N}\}$ and

$$\sum_{n=1}^{\infty} \{1 - [m_1(\{a_n\})/m_2(\{a_n\})]^{1/\alpha}\}^2 < \infty.$$

Furthermore if any of these conditions fail, μ_{Z_1} and μ_{Z_2} are singular.

(Note that condition iii) is symmetric in m_1 and m_2 and independent of α as $\sum_n (1 - x_n)^2 < \infty$ if and only if $\sum_n (1 - x_n^q)^2 < \infty$).

Proof. First suppose that m_1 and m_2 are not equivalent, e.g. $m_2 \not\ll m_1$. Then there exists $B \in \sigma(\mathfrak{I})$ such that

$$\|Z_1(B)\|_{\alpha_1}^{\alpha_1} = m_1(B) = 0, \quad \text{and} \quad \|Z_2(B)\|_{\alpha_2}^{\alpha_2} = m_2(B) > 0.$$

It follows that Z_1 does not dominate Z_2 and by Proposition 3.2.2 we have singularity. From now on we assume $m_1 \sim m_2$.

Suppose $\alpha_1 \neq \alpha_2$. Since m_1 and m_2 are not purely atomic with a finite number of atoms, we can choose an infinite sequence $(B_n; n \in \mathbf{N})$ of disjoint sets in $\mathfrak{I}$ such that $m_i(B_n) > 0$, i=1.2. Define $\Psi: X^{\mathfrak{I}} \to X^{\mathbf{N}}$ by

$$\Psi(x) = (\Psi_n(x) = x(B_n); \ n \in \mathbb{N}).$$

Thus, for i=1,2, under μ_{Z_i}, Ψ is a sequence of independent $S\alpha_i S$ random variables with $\|\Psi_n\|_{\alpha_i}^{\alpha_i} = m_i(B_n)$. It follows from Proposition 3.1.5 that if $\alpha_1 \neq \alpha_2$, then $\mu_{Z_i}\Psi^{-1} \perp \mu_{Z_2}\Psi^{-1}$, so that $\mu_{Z_1} \perp \mu_{Z_2}$. From now on we assume $\alpha_1 = \alpha_2 = \alpha$.

Since $m_1 \sim m_2$ we have $m_{1,d} \sim m_{2,d}$. Suppose $m_{1,d} \neq m_{2,d}$, so that

$$m_{i,d}(\{dm_{2d}/dm_{1d} \neq 1\}) > 0, \quad i=1,2,$$

hence

$$m_{i,d}(\{0 < dm_{2,d}/dm_{1,d} < 1\}) > 0 \text{ or } m_{i,d}(\{dm_{2d}/dm_{1,d} > 1\}) > 0.$$

Assume $m_{i,d}(\{dm_{2,d}/dm_{1,d} > 1\}) > 0$. Then there exists $\delta > 1$ such that $m_{i,d}(\{dm_{2,d}/dm_{1,d} > \delta\}) > 0$. Since $m_{i,d}$ is nonatomic, we can find a sequence $(B_n; \ n \in \mathbb{N})$ of disjoint subsets of $\{dm_{i,d}/dm_{1,d} > \delta\}$ such that $m_{1,d}(B_n) > 0$. Let $\Phi: X^{\mathfrak{J}} \to X^{\mathbb{N}}$ be the map defined by

$$\Phi(x) = (\Phi_n(x) = x(A^c \cap B_n)/m_{1,d}(B_n)^{1/\alpha}; \ n \in \mathbb{N}).$$

Under μ_{Z_1}, Φ is an i.i.d. sequence of standard $S\alpha S$ r.v.'s, and under μ_{Z_2}, Φ is an independent sequence of $S\alpha S$ r.v.'s with

$$\|\Phi_n\|_\alpha^\alpha = m_{2,d}(B_n)/m_{1,d}(B_n).$$

It follows from Corollary 3.1.4 and Kanter (1975) that $\mu_{Z_1}\Sigma^{-1}$ and $\mu_{Z_2}\Sigma^{-1}$ are either equivalent or singular, and they are singular if and only if

$$(3.3.1) \qquad \sum_{n=1}^{\infty} [1 - (m_{2,d}(B_n)/m_{1,d}(B_n))^{1/\alpha}]^2 = \infty.$$

Now by construction

$$m_{2,d}(B_n) = \int_{B_n} \frac{dm_{2,d}}{dm_{1,d}}\, dm_{1,d} > \delta\, m_{1,d}(B_n).$$

Hence $1 < \delta < m_{2,d}(B_n)/m_{1,d}(B_n)$, so that (3.3.1) holds. Thus $\mu_{Z_1}\Sigma^{-1} \perp \mu_{Z_2}\Sigma^{-1}$ which implies $\mu_{Z_1} \perp \mu_{Z_i}$.

If $m_{i,d}(\{dm_{2,d}/dm_{1,d} > 1\}) = 0$ we have $m_{i,d}(\{dm_{1,d}/dm_{2,d} > 1\}) > 0$ and an identical argument applies. Therefore $m_1 \sim m_2$ and $m_{1,d} \neq m_{2,d}$ implies $\mu_{Z_1} \perp \mu_{Z_2}$.

Now assume $m_{1,d} = m_{2,d}$. Since $m_1 \sim m_2$, they have the same set of atoms $A = \{a_n;\, n \in \mathbf{N}\}$. Suppose $\mu_{Z_2} \ll \mu_{Z_1}$ and let $\Xi \colon \mathbf{X}^J \to \mathbf{X}^{\mathbf{N}}$ be defined by

$$\Xi(x) = (\Xi_n(x) = x(\{a_n\})/m_1(\{a_n\})^{1/\alpha};\, n \in \mathbf{N}) .$$

Thus $\mu_{Z_2}\Xi^{-1} \ll \mu_{Z_1}\Xi^{-1}$ and Ξ is an i.i.d. sequence of standard SαS r.v.'s under μ_{Z_1} and under μ_{Z_2} an independent sequence of SαS r.v.'s with $\|\Xi\|_\alpha^\alpha = m_2(\{a_n\})/m_1(\{a_n\})$. Hence by Corollary 3.1.4 and Kanter (1975),

$$(3.3.2) \qquad \sum_{n=1}^{\infty}[1 - (m_2(\{a_n\})/m_1(\{a_n\}))^{1/\alpha}] < \infty.$$

Also, if (3.3.2) does not hold, again Corollary 3.1.4 and Kanter (1975) imply $\mu_{Z_1}\Xi^{-1} \perp \mu_{Z_2}\Xi^{-1}$ so that $\mu_{Z_1} \perp \mu_{Z_2}$.

Conversely, suppose that i), ii) and iii) hold. Since $m_{1,d} = m_{2,d}$ we have

$$Z_i \overset{\mathrm{L}}{=} Z_{i,a} + Z_d, \quad i = 1,2,$$

where $Z_{i,a}$ and Z_d are independent, independently scattered SαS random measures with control measures $m_{i,a}$ and $m_d = m_{1,d} = m_{2,d}$ respectively. Let $\Phi: \mathbf{X}^\mathbf{N} \to \mathbf{X}^\mathbf{J}$ be defined by

$$[\Phi(y)](B) = \Phi(y,B) = \sum_{n=1}^\infty 1_B(a_n) m_1(\{a_n\})^{1/\alpha} y_n, \quad y = (y_n) \in \mathbf{X}^\mathbf{N}.$$

Thus $(\Phi \circ \Xi)(Z_i) \overset{\mathbf{L}}{=} Z_{i,a}$, so that $\mu_{Z_{i,a}} = (\mu_{Z_i} \Xi^{-1}) \Phi^{-1}$, $i=1,2$. Now by Corollary 3.1.4, iii) implies $\mu_{Z_1} \Xi^{-1} \sim \mu_{Z_2} \Xi^{-1}$, hence $\mu_{Z_{1,a}} \sim \mu_{Z_{2,a}}$. Therefore, since $\mu_{Z_i} = \mu_{Z_{i,a}} * \mu_{Z_d}$, $i=1,2$, it follows that $\mu_{Z_1} \sim \mu_{Z_2}$. $\qquad\qquad\square$

Remark: It also follows from the proof of Proposition 3.3.1 that if only one of the control measures has a finite number of atoms μ_{X_1} and μ_{X_2} are singular.

As in the case of admissible translates the results on equivalence and singularity of independently scattered SαS random measures can be extended to certain invertible SαS processes.

Let $X_i = (X_i(t); t \in \mathbf{T})$, $i=1,2$, be two invertible SαS processes with spectral representations $X_i(t) = \int_\mathbf{I} f(t,u) Z_i(du)$ and control measures m_i where $f \in L_{\alpha_1}(m_1) \cap L_{\alpha_2}(m_2)$ and the independently scattered random measures have the same δ-ring $\mathbf{J}$ of subsets of $\mathbf{I}$ as parameter space, i.e., m_1 and m_2 have the same sets of finite measure. X_1 and X_2 will be said to be simultaneously invertible if for each $B \in \mathbf{J}$ there exist $N_n(B)$, $a_{n,1}(B), ..., a_{n,N_n(B)}(B)$, $t_{n,1}(B), ..., t_{n,N_n(B)}(B)$ such that

$$\sum_{k=1}^{N_n(B)} a_{n,k}(B) f(t_{n,k}(B), \cdot) \quad \to \quad 1_B(\cdot) \qquad\qquad \text{as } n \to \infty.$$

in $L_{\alpha_i}(m_i)$ for both $i=1,2$. E.g., X_1 and X_2 are simultaneously invertible if they are invertible, $\alpha_1 = \alpha_2$ and dm_1/dm_2 is bounded above or below. In particular X_1 and X_2

are simultaneous invertible if they are invertible and their associated random measures Z_1 and Z_2 are equivalent, as $\mu_{Z_1} \sim \mu_{Z_2}$ implies $\alpha_1 = \alpha_2$, $m_{1,d} = m_{2,d}$ and $\sum_{n=1}^{\infty} \left(1 - \left(dm_1 / dm_2(a_n)\right)^{1/\alpha}\right)^2 < \infty$.

The simultaneous invertibility of X_1 and X_2 allows for the study of the Lebesgue decomposition of μ_{X_1} with respect to μ_{X_2} in terms of the decomposition of μ_{Z_1} with respect to μ_{Z_2}. Indeed $X_i(t) = \int f(t,u) \, Z_i(du)$ is roughly speaking $X_i = L(Z_i)$, where L is a linear map from $\mathcal{L}(Z)$ into $\mathcal{L}(X)$, so we expect the singularity of X_1 and X_2 to imply the singularity of Z_1 and Z_2, and conversely the equivalence of Z_1 and Z_2 to imply the equivalence of X_1 and X_2. Simulatenous invertibility is like having $Z_i = L^{-1}(X_i)$, so we should have the singularity of Z_1 and Z_2 implying the singularity of X_1 and X_2, and conversely the equivalence of X_1 and X_2 implying that of Z_1 and Z_2. Hence with both we expect to have the above implications in both directions. The next proposition makes this precise.

<u>Proposition</u> <u>3.3.2</u>. Let $X_i = (X_i(t); t \in \mathbb{T})$ be two simultaneously invertible $S\alpha_i S$ processes with spectral representations $Y_i(t) = \int_{\mathbb{I}} f(t,u) \, Z_i(du)$ and control measures m_i which are not purely discrete with a finite number of atoms. Then μ_{X_1} and μ_{X_2} are either equivalent or singular, and

i) $\quad \mu_{X_1} \sim \mu_{X_2} \quad \Leftrightarrow \quad \mu_{Z_1} \sim \mu_{Z_2}$.

ii) $\quad \mu_{X_1} \perp \mu_{X_2} \quad \Leftrightarrow \quad \mu_{Z_1} \perp \mu_{Z_2}$,

i.e. $\mu_{X_1} \sim \mu_{X_2}$ if and only if conditions i), ii) and iii) of Proposition 3.3.1 are satisfied, and $\mu_{X_1} \perp \mu_{X_2}$ if and only if at least one of these conditions fail.

<u>Proof</u>: As in Proposition 2.2.3 for $B \in \mathbb{J}$ we can define

$$\Phi_n(B,x) = \sum_{k=1}^{N_n(B)} a_{n,k}(B) x(t_{n,k}(B)), \quad x \in \mathbf{X}^{\mathbb{T}}.$$

so that

$$\Phi_n(B, X_i(\cdot, \omega)) \to Z_i(B, \omega), \quad i = 1, 2,$$

in probability as $n \to \infty$. Let $(\Phi_{n_k}(B, \cdot); k \in \mathbb{N})$ be a subsequence converging a.e. (μ_{X_i}), $i = 1, 2$, and put

$$\tilde{Z}(B) = \tilde{Z}(B, \cdot) = \liminf_{k \to \infty} \Phi_{n_k}(B, \cdot) 1_{\{x; \Phi_{n_k}(x) \text{ converges}\}}(\cdot) .$$

Hence

$$\tilde{Z}(B, X_i(\cdot, \omega)) = Z_i(B, \omega) \quad \text{a.s.}, \quad i = 1, 2.$$

The stochastic process $\tilde{Z} = (\tilde{Z}(B), B \in \mathfrak{J})$ defined on $(\mathbf{X}^{\mathsf{T}}, \mathcal{C})$ is an independently scattered SαS random measure with control measure m_1 under μ_{X_1} and m_2 under μ_{X_2}. If we also denote by $\tilde{Z}$ the map $x \to \tilde{Z}(\cdot, x)$ then

$$\mu_{X_1} \sim \mu_{X_2} \quad \Rightarrow \quad \mu_{X_1}\tilde{Z}^{-1} \sim \mu_{X_2}\tilde{Z}^{-1} \quad (\text{i.e. } \mu_{Z_1} \sim \mu_{Z_2})$$

and

$$\mu_{Z_1} \perp \mu_{Z_2} \quad (\text{i.e. } \mu_{X_1}\tilde{Z}^{-1} \perp \mu_{X_2}\tilde{Z}^{-1}) \quad \Rightarrow \quad \mu_{X_1} \perp \mu_{X_2}.$$

On the other hand if $\mu_{Z_1} \sim \mu_{Z_2}$, i.e. $\mu_{X_1}\tilde{Z}^{-1} \sim \mu_{X_2}\tilde{Z}^{-1}$, it follows that i), ii) and iii) of Proposition 3.3.1 hold. Thus, we can construct independent processes $\tilde{X}_d$ and $\tilde{X}_{i,a}$ on $(\mathbf{X}^{\mathfrak{J}}, \mathcal{C}(\mathbf{X}^{\mathfrak{J}}), \mu_{Z_i})$ such that

$$X_i \stackrel{\mathsf{L}}{=} \tilde{X}_d + \tilde{X}_{i,a}, \quad i = 1, 2,$$

with $\mu_{\tilde{X}_{1,a}} \sim \mu_{\tilde{X}_{2,a}}$. Since $\mu_{X_i} = \mu_{\tilde{X}_d} * \mu_{\tilde{X}_{i,a}}$ we have $\mu_{X_1} \sim \mu_{X_2}$.

Now if μ_{X_1} and μ_{X_2} are not equivalent it follows that $\mu_{Z_1} \perp \mu_{Z_2}$ (since otherwise $\mu_{Z_1} \sim \mu_{Z_2}$, which implies $\mu_{X_1} \sim \mu_{X_2}$, i.e. a contradiction) and this was shown to imply $\mu_{X_1} \perp \mu_{X_2}$. $\square$

It follows from Proposition 3.3.2 that simultaneously invertible processes are singular whenever their indexes of stability are different. This is not generally true for symmetric stable processes with different indexes of stability. Indeed, let $G = (G(t); t \in T)$ be a Gaussian processes, and for $i=1,2$, A_i a standard positive $(\alpha_i/2)$-stable random variable with $\alpha_1 \neq \alpha_2$, and consider the sub-Gaussian $S\alpha_i S$ processes

$$X_i = (X_i(t) = A_i^{1/2}G(t); t \in T).$$

We have that

$$\mu_{X_i}(B) = \int_{\mathbb{R}^+} \mu_{xG}(B) \, \mu_{A_i}(dx).$$

Since the distribution μ_{A_i} of A_i has positive density in $\mathbb{R}^+$ we have $\mu_{A_1} \sim \mu_{A_2}$, so that by the Corollary of Theorem 18.1 in Skorohod (1974), $\mu_{X_1} \sim \mu_{X_2}$. Since the linear space of sub-Gaussian processes does not contain (nondegenerate) independent random variables, sub-Gaussian processes are not invertible (nor simultaneously invertible). Further examples of symmetric stable processes with different indexes of stability which are equivalent are

$$X_i = (X_i(t) = \sum_{n=1}^{N} A_{i,n}^{1/2} G_n(t); t \in T)$$

where for each $i=1,2$, the vector $(A_{i,1}, ..., A_{i,N})$ is positive $(\alpha_i/2)$-stable, independent of the mutually independent Gaussian processes $G_n = (G_n(t); t \in T)$, $n=1,...,N$.

Next Proposition 3.3.2 is applied to describe the Lebesgue decomposition between two $S\alpha S$ harmonizable processes, and to show that muliples of invertible processes are singular.

<u>Corollary 3.3.3</u>. Let $X_i = (X_i(t); t \in T)$ be two harmonizable $S\alpha_i S$ processes, i.e.

$$X_i(t) = \int_{\mathbf{I}} \exp(i<t, u>)Z_i(du), \quad t \in T,$$

where $\mathbf{I} = \mathbf{R}^d$ and $[-\pi, \pi]^d$ for $T = \mathbf{R}^d$ and $\mathbf{Z}^d$ respectively, with spectral measures m_i not purely discrete with a finite number of atoms, $i=1,2$. Then μ_{X_1} and μ_{X_2} are equivalent if and only if

i) $\alpha_1 = \alpha_2 = \alpha$,

ii) $m_{1,d} = m_{2,d}$,

iii) m_1 and m_2 have the same atoms $\{a_n; n \in \mathbf{N}\}$ with

$$\sum_{n=1}^{\infty} [1 - m_1(\{a_n\})/m_2(\{a_n\})]^2 < \infty,$$

and they are singular otherwise.

<u>Proof</u>: Clearly X_1 and X_2 are simulatenously invertible, since indicator functions can be approximated uniformly by linear combinations of the functions $f(t,u) = \exp(i<t,u>)$. Hence the result follows from Proposition 3.3.2.   □

<u>Corollary 3.3.4</u>. Let $X = (X(t); t \in T)$ be an invertible $S\alpha S$ process with control measure m which is not purely atomic with a finite number of atoms. Then μ_X and μ_{bX} are singular wherever $|b| \neq 1$.

<u>Proof</u>. If $X(t) = \int_{\mathbf{I}} f(t,u) Z(du)$, where Z has control measure m, then

$$(bX)(t) = bX(t) = \int bf(t,u) Z(du) = \int f(t,u) Z_b(du)$$

where Z_b has control measure $|b|^\alpha m$. Cleary X and bX are simultaneously invertible and since m is not purely atomic with a finite number of atoms, the result follows from Proposition 3.3.2. $\qquad \Box$

The result in Corollary 3.3.4 is known to hold for every Gaussian processes with infinite dimensional linear space. Here again the class of SαS sub-Gaussian processes provides an example to show that the result is not true for all infinite dimensional SαS processes. In fact, if $G=(G(t); t \in \mathbb{T})$ is Gaussian, A is a standard positive $(\alpha/2)$-stable random variable and $X(t) = (A^{1/2}G(t), t \in \mathbb{T})$, reasoning as in page 73 we have for each $b > 0$,

$$\mu_{bX}(B) = \int_{\mathbb{R}^+} \mu_{xG}(B) \, \mu_{bA}(dx).$$

The distributions μ_A and μ_{bA} are equivalent for all $b > 0$ so that $\mu_X \sim \mu_{bX}$.

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